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ORIGINAL STUDY

Weakly Pseudo Primary 2-absorbing Submodules

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Abstract

In this paper, we introduce the notion of a weakly pseudo primary 2-absorbing sub-module as a generalization of a 2-absorbing sub-module and a pseudo 2-absorbing sub-module. Moreover, we give many basic properties, examples, and characterizations of these notions.

Subj-class something like: 13A05, 13C13, 13C60, 13C10, 13E05, 13F15

Keywords: Primary- 2-absorbing, Pseudo -primary- 2-absorbing, 2-absorbing

1. Introduction

In this paper, all rings are commutative with $0\neq 1$, and D is a unitary R-module. During the last 13 years, the notion of 2-absorbing sub-modules and weakly 2-absorbing sub-modules has been previously investigated by Darani and Soheilinia [1], and its generalizations to where 2-absorbing ideal was first introduced in 2007 by Badawi [2]. Let $A\subsetneq D$ be sub-module of R-module D, A is called 2-absorbing (weakly 2-absorbing) if whenever $r_1r_2d\in A$ ($0\neq r_1r_2d\in A$) for some $r_1, r_2\in R$, $d\in D$, then $r_1d\in A$ or $r_2d\in$ A or $r_1r_2\in[A:_RD]$. Following that, Mostafanasab, Yetkin, Tekir and Daran [3] introduced the concept primary 2-absorbing it's generalizations of primary sub-module and weakly primary sub-module, respectively, a sub-module $A\subsetneq D$ of an $R-$module D is called a primary 2-absorbing, if $r_1r_2d\in A$, for $r_1, r_2\in R$, $d\in D$, implies that either $r_1d\in rad_D(A)$ or $r_2d\in rad_D(A)$ or $r_1r_2\in[A + Soc(D):_RD]$. Also, Mohammadali and Abdulla [4] introduced the concept of pseudo primary 2-absorbing sub-module, a sub-module $A\subsetneq D$ of an $R-$module D, is called a pseudo primary 2-absorbing sub-module D, if $r_1r_2d\in A$, for $r_1, r_2\in R$, $d\in D$, implies that either $r_1d\in rad_D(A)+Soc(D)$ or $r_2d\in rad_D(A)+Soc(D)$ or $r_1r_2\in[A + Soc(D):_RD]$. In earlier years, there has been prior research that has addressed the following themes: pseudo-2-absorbing, pseudo semi 2-absorbing [5], and weakly pseudo semi 2-absorbing [6]. This paper is based on introducing a new class of sub-modules called weakly pseudo primary 2-absorbing sub-modules. Among many other results in this paper, we show in Proposition 2 and Remark 3 the relation between the above concepts with weakly pseudo primary 2-absorbing sub-module with examples. Also, we show that the intersection of each pair of distinct weakly pseudo primary 2-absorbing sub-modules is not a weakly pseudo primary 2-absorbing sub-module in general. Also, the residual of a weakly pseudo primary 2-absorbing sub-module is a weakly pseudo primary 2-absorbing ideal. To do this, specific requirements will be conditions. In Theorem 14, we prove A is a weakly pseudo primary 2-absorbing sub-module of an R-module D if and only if $(0)\neq Q_1Q_2T\subseteq A$ for some ideal Q_1, Q_2 of R and sub-module T of D, implies that either $Q_1T\subseteq rad_D(A)+Soc(D)$ or $Q_2T\subseteq rad_D(A)+Soc(D)$ or $Q_1Q_2\subseteq[A + Soc(D):_RD]$.

2. Weakly pseudo primary 2-absorbing sub-module

Definition 1. Let $A\subsetneq D$ be a sub-module of an $R-$module D, we said A be a weakly pseudo primary 2-absorbing sub-module D, if $0\neq r_1r_2d\in A$, for r_1,

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$r_2 \in R$, $d \in D$, implies that either $r_1 d \in rad_D(A) + Soc(D)$ or $r_2 d \in rad_D(A) + Soc(D)$ or $r_1 r_2 \in [A + Soc(D):_R D]$. An ideal Q of a ring R is said to be a weakly pseudo primary 2-absorbing ideal of R if Q is a weakly pseudo primary 2-absorbing sub-module an $R-$module R.

Proposition 2. *Let $A \subsetneq D$ be a sub-module of an R-module D, then the following holds.*

1. *If A is a 2-absorbing (weakly 2-absorbing) sub-module of D, Then A is a weakly pseudo primary 2-absorbing sub-module of, generally, the converse need not hold.*
2. *If A is a pseudo-2-absorbing (weakly pseudo-2-absorbing) sub-module of D, Then A is a weakly pseudo primary 2-absorbing sub-module of D, generally, the converse need not hold.*
3. *If A is a primary 2-absorbing (weakly primary 2-absorbing) sub-module of D, Then A is a weakly pseudo primary 2-absorbing sub-module of D, generally, the converse need not hold.*
4. *If A is a pseudo primary 2-absorbing sub-module of D, Then A is a weakly pseudo primary 2-absorbing sub-module of D, generally, the converse need not hold.*

Proof.

1. Assume that A is 2-absorbing. To prove A is a weakly pseudo primary 2-absorbing. Let $0 \neq r_1 r_2 d \in A$ for some $r_1, r_2 \in R$ and $d \in D$, since A is 2-absorbing, then ether $r_1 d \in A \subseteq rad_D(A) \subseteq rad_D(A) + Soc(D)$ or $r_2 d \in A \subseteq rad_D(A) \subseteq rad_D(A) + Soc(D)$ or $r_1 r_2 D \subseteq A \subseteq A + Soc(D)$. Now we have ether $r_1 d \in rad_D(A) + Soc(D)$ or $r_2 d \in rad_D(A) + Soc(D)$ or $r_1 r_2 \in [A + Soc(D):_R D]$. Thus A is a weakly pseudo primary 2-absorbing. In regards to the contrary, the $12\mathbb{Z}$ is a proper sub-module of $\mathbb{Z}$-module $\mathbb{Z}$, $12\mathbb{Z}$ is weakly pseudo primary 2-absorbing sub-module of $\mathbb{Z}$-module $\mathbb{Z}$, since for all $0 \neq r_1 r_2 d \in 12\mathbb{Z}$ for r_1, r_2, $d \in \mathbb{Z}$, implies that either $r_1 d \in rad_{\mathbb{Z}}(12\mathbb{Z}) + Soc(\mathbb{Z}) = 6\mathbb{Z} + (0) = 6\mathbb{Z}$ or $r_2 d \in rad_{\mathbb{Z}}(12\mathbb{Z}) + Soc(\mathbb{Z}) = 6\mathbb{Z}$ or $r_1 r_2 \in [12\mathbb{Z} + Soc(\mathbb{Z}):_{\mathbb{Z}} \mathbb{Z}] = 12\mathbb{Z}$. For example, $0 \neq 2.2.3 \in 12\mathbb{Z}$ for $2, 3 \in \mathbb{Z}$, implies that $2.3 = 6 \in 6\mathbb{Z}$ but $2.2 = 4 \notin 12\mathbb{Z}$. But $12\mathbb{Z}$ is not 2-absorbing sub-module of $\mathbb{Z}$-module $\mathbb{Z}$ since $2.2.3 \in 12\mathbb{Z}$ for $2, 3 \in \mathbb{Z}$, but $2.3 = 6 \notin 12\mathbb{Z}$ and $2.2 = 4 \notin 12\mathbb{Z}$.
2. Assume that A is pseudo-2-absorbing. To prove A is weakly pseudo primary 2-absorbing. Let $0 \neq r_1 r_2 d \in A$ for some $r_1, r_2 \in R$ and $d \in D$, since A is pseudo-2-absorbing, then ether $r_1 d \in A + Soc(D) \subseteq rad_D(A) + Soc(D)$ or $r_2 d \in A + Soc(D) \subseteq rad_D(A) + Soc(D)$ or $r_1 r_2 D \subseteq A + Soc(D)$. Now we have ether $r_1 d \in rad_D(A) + Soc(D)$ or $r_2 d \in rad_D(A) + Soc(D)$ or $r_1 r_2 \in [A + Soc(D):_R D]$. Thus A is a weakly pseudo primary 2-absorbing. In regards to the contrary, let $D = \mathbb{Z}$, $R = \mathbb{Z}$, and $A = 16\mathbb{Z}$. A is proper sub-module of D, A is weakly pseudo primary 2-absorbing, since for all $0 \neq r_1 r_2 d \in A$ for $r_1, r_2 \in R$, and $d \in D$ implies that either $r_1 d \in rad_D(A) + Soc(D) = 2\mathbb{Z} + (0) = 2\mathbb{Z}$ or $r_2 d \in rad_D(A) + Soc(D) = 2\mathbb{Z}$ or $r_1 r_2 \in [A + Soc(D):_R D] = 16\mathbb{Z}$. For example, $2.3.8 = 48 \in A$ for $2, 3 \in R$, $8 \in D$, implies that $2.8 = 16 \in 2\mathbb{Z}$ but $2.3 = 6 \notin 16\mathbb{Z}$. But A is not pseudo-2-absorbing. Since $2.2.4 \in 16\mathbb{Z}$, where $2 \in R$ and $4 \in D$, but $2.4 = 8 \notin A + Soc(D) = 16\mathbb{Z} + (0) = 16\mathbb{Z}$ and $2.2 = 4 \notin [16\mathbb{Z} + Soc(D):_R D] = 16\mathbb{Z}$.
3. Assume that A is a primary 2-absorbing sub-module of R-module D. Let $o \neq r_1 r_2 d \in A$ for some $r_1, r_2 \in R$ and $d \in D$, since A is a primary 2-absorbing, then either $r_1 d \in rad_D(A) \subseteq rad_D(A) + Soc(D)$ or $r_2 d \in rad_D(A) \subseteq rad_D(A) + Soc(D)$ or $r_1 r_2 D \subseteq A \subseteq A + Soc(D)$. That is either $r_1 d \in rad_D(A) + Soc(D)$ or $r_2 d \in rad_D(A) + Soc(D)$ or $r_1 r_2 \in [A + Soc(D):_R D]$. Hence A is a weakly pseudo primary 2-absorbing sub-module of R-module D. In regards to the contrary, In $\mathbb{Z}$ -module $\mathbb{Z}_{90}$, the $(\overline{30})$ is a proper sub-module of $\mathbb{Z}$ -module $\mathbb{Z}_{90}$. $(\overline{30})$ is weakly pseudo primary 2-absorbing sub-module since for all $r_1, r_2 \in \mathbb{Z}$ and $d \in \mathbb{Z}_{90}$ with $0 \neq r_1 r_2 d \in (\overline{30})$ we have either $r_1 d \in rad_{\mathbb{Z}_{90}}((\overline{30})) + Soc(\mathbb{Z}_{90}) = (\overline{30}) + (\overline{3}) = (\overline{3})$ or $r_2 d \in rad_{\mathbb{Z}_{90}}((\overline{30})) + Soc(\mathbb{Z}_{90}) = (\overline{3})$ or $r_1 r_2 \in [(\overline{30}) + Soc(\mathbb{Z}_{90}):_{\mathbb{Z}} \mathbb{Z}_{90}] = 3\mathbb{Z}$. Since $Soc(\mathbb{Z}_{90}) = (\overline{3})$ and $rad_D((\overline{30})) = (\overline{30})$. For example, $0 \neq 2.3.\overline{5} = \overline{30} \in (\overline{30})$, we have $3.\overline{5} = \overline{15} \in rad_{\mathbb{Z}_{90}}((\overline{30})) + Soc(\mathbb{Z}_{90}) = (\overline{30}) + (\overline{3}) = (\overline{3})$ and $2.3 = 6 \in 3\mathbb{Z}$.but $2.\overline{5} = \overline{10} \notin (\overline{3})$. It is clear that $(\overline{30})$ is not primary 2-absorbing sub-module since $2.3.\overline{5} = \overline{30} \in (\overline{30})$, $3.\overline{5} = \overline{15} \notin (\overline{30})$ and $2.\overline{5} = \overline{10} \notin (\overline{30})$ and $2.3 = 6 \notin 30\mathbb{Z}$.
4. Clear by Definition of weakly pseudo primary 2-absorbing sub-module.

Remark 3. The relation between the semi-2-absorbing (weakly semi-2-absorbing, pseudo-semi-2-absorbing and weakly pseudo-semi-2-absorbing) sub-modules with weakly pseudo primary 2-absorbing sub-module is independent. That is

1. If A is a semi-2-absorbing (weakly semi-2-absorbing) sub-module of R-module D, it is not necessarily to be a weakly pseudo primary 2-absorbing sub-module of R-module D.
2. If A be a weakly pseudo primary 2-absorbing sub-module of R-module D, it is not necessarily

to be semi-2-absorbing (weakly semi-2-absorbing) sub-module of R-module D.
3. If A is a pseudo-semi-2-absorbing (weakly pseudo-semi-2-absorbing) sub-module of R-module D, it is not necessarily a weakly pseudo primary 2-absorbing sub-module of R-module D.
4. If A is a weakly pseudo primary 2-absorbing sub-module of R-module D, it is not necessarily a pseudo-semi-2-absorbing (weakly pseudo-semi-2-absorbing) sub-module of R-module D.

The following Example 4 clear that.

Example 4.

1. Assume that $42\mathbb{Z}$ is proper sub-module of $\mathbb{Z}$ -module $\mathbb{Z}$. $42\mathbb{Z}$ is semi-2-absorbing (weakly semi-2-absorbing)sub-module, since for all $r^2d\in 42\mathbb{Z}$, $r, d\in\mathbb{Z}$, implies that either $rd\in 42\mathbb{Z}$ or $r^2\in [42\mathbb{Z}:_{\mathbb{Z}}\mathbb{Z}] = 42\mathbb{Z}$. For example, $2^2.21 = 84\in 42\mathbb{Z}$, $2, 21\in\mathbb{Z}$ implies that $2.21 = 42\in 42\mathbb{Z}$ but $2^2 = 4\notin[42\mathbb{Z}:_{\mathbb{Z}}\mathbb{Z}] = 42\mathbb{Z}$. But $42\mathbb{Z}$ not weakly pseudo primary 2-absorbing sub-module, since $0\neq 2.3.7 = 42\in 42\mathbb{Z}$ for $2, 3, 7\in\mathbb{Z}$. But $2.7 = 14\notin rad_{\mathbb{Z}}(42\mathbb{Z}) + Soc(\mathbb{Z}) = 42\mathbb{Z} + (0) = 42\mathbb{Z}$, and $3.7 = 21\notin rad_{\mathbb{Z}}(42\mathbb{Z}) + Soc(\mathbb{Z}) = 42\mathbb{Z}$ and $2.3 = 6\notin [42\mathbb{Z} + Soc(\mathbb{Z}):_{\mathbb{Z}}\mathbb{Z}] = 42\mathbb{Z}$.
2. Assume that $16\mathbb{Z}$ is proper sub-module of $\mathbb{Z}$ -module $\mathbb{Z}$. $16\mathbb{Z}$ is weakly pseudo primary 2-absorbing sub-module (by Proposition 2 [2]). But it is not a semi-2-absorbing (weakly semi-2-absorbing) sub-module, since $2^2.4 = 16\in 16\mathbb{Z}$ for $2,4\in\mathbb{Z}$, but $2.4 = 8\notin 16\mathbb{Z}$ and $2^2 = 4\notin[16\mathbb{Z}:_{\mathbb{Z}}\mathbb{Z}] = 16\mathbb{Z}$.
3. Similar [1] since $Soc(\mathbb{Z}) = 0$.
4. Similar [2] since $Soc(\mathbb{Z}) = 0$.

Lemma 5. [1] *On $m\mathbb{Z}$ is a 2-absorbing sub-module of $\mathbb{Z} -$ module $\mathbb{Z}$ if $m = 0$ or $m = p_1$ or $m = p_1p_2$ where p_1, p_2 are prime integers.*

Proposition 6. *$m\mathbb{Z}$ is a weakly pseudo primary 2-absorbing of $\mathbb{Z} -$ module $\mathbb{Z}$ if $m = p_1$, $\mathrm{m} = {p_1}^2$, $\mathrm{m} = p_1p_2$, where p_1, p_2 are prime integers.*

Proof. Since $n\mathbb{Z}$ is a 2-absorbing sub-module by Lemma 5 and since every 2-absorbing sub-module is weakly pseudo primary 2-absorbing by Proposition 2. Then $n\mathbb{Z}$ weakly pseudo primary 2-absorbing.

Remark 7. Let A_1 and A_2 are two distinct weakly pseudo primary 2-absorbing sub-modules of an R-module D. Then $A_1\cap A_2$ is not necessarily weakly pseudo primary 2-absorbing. the following Example 8 clear that.

Example 8. Let $D = \mathbb{Z}$, $R = \mathbb{Z}$, $A_1 = 7\mathbb{Z}$, $A_2 = 6\mathbb{Z}$ are two weakly pseudo primary 2-absorbing sub-modules of $\mathbb{Z}$-module $\mathbb{Z}$ by Proposition 6 But $A_1\cap A_2 = 42\mathbb{Z}$ is not weakly pseudo primary 2-absorbing, by *Example 4* [1].

Remark 9. The residual of a weakly pseudo primary 2-absorbing sub-module of an R-module D is not necessarily to be a weakly pseudo primary 2-absorbing ideal of R. The following Example 10 clear that.

Example 10. $(\bar{0})$ is a proper sub-module of $\mathbb{Z}$ -module $\mathbb{Z}_{30}$. $(\bar{0})$ is a weakly pseudo primary 2-absorbing, by definition. But $[(\bar{0}):_{\mathbb{Z}}\mathbb{Z}_{30}] = 30\mathbb{Z}$ is not weakly pseudo primary 2-absorbing ideal of $\mathbb{Z}$. Since $2.3.5 = 30\in 30\mathbb{Z}$ for $2, 3, 5\in\mathbb{Z}$. But $2.5 = 10\notin rad_D(30\mathbb{Z}) + Soc(\mathbb{Z}) = 30\mathbb{Z} + (0) = 30\mathbb{Z}$ and $3.5 = 15\notin rad_D(30\mathbb{Z}) + Soc(\mathbb{Z}) = 30\mathbb{Z}$ and $2.3 = 6\notin[30\mathbb{Z} + Soc(\mathbb{Z}):_{\mathbb{Z}}\mathbb{Z}] = 30\mathbb{Z}$.

Proposition 11. *Let $A\subsetneq D$ be a sub-module of R-module D. A is weakly pseudo primary 2-absorbing if and only if for any $r_1, r_2\in R$ such that $r_1r_2\notin[A + Soc(D):_RD]$ we have $[A:_Dr_1r_2]\subseteq[0:_Dr_1r_2]\cup[rad_D(A) + Soc(D):_Dr_1]\cup[rad_D(A) + Soc(D):_Dr_2]$.*

Proof. Assume that A is weakly pseudo primary 2-absorbing sub-module and let $x\in[A:_Dr_1r_2]$ to prove $x\in[0:_Dr_1r_2]\cup[rad_D(A) + Soc(D):_Dr_1]\cup[rad_D(A) + Soc(D):_Dr_2]$. Then $r_1r_2x\in A$. If $0\neq r_1r_2x\in A$ and $r_1r_2\notin[A + Soc(D):_RD]$, it follows that $r_1x\in rad_D(A) + Soc(D)$ or $r_2x\in rad_D(A) + Soc(D)$ (since A is weakly pseudo primary 2-absorbing). Thus either $x\in[rad_D(A) + Soc(D):_Dr_1]$ or $x\in[rad_D(A) + Soc(D):_Dr_2]$. If $0 = r_1r_2x\in A$ then $x\in[0:_Dr_1r_2]$. Hence $x\in[0:_Dr_1r_2]\cup[rad_D(A) + Soc(D):_Dr_1]\cup[rad_D(A) + Soc(D):_Dr_2]$ That is $[A:_Dr_1r_2]\subseteq[0:_Dr_1r_2]\cup[rad_D(A) + Soc(D):_Dr_1]\cup[rad_D(A) + Soc(D):_Dr_2]$.
Conversely, Assume that $0\neq r_1r_2d\in A$, for $r\in R$, $d\in D$ and let $[A:_Dr_1r_2]\subseteq[0:_Dr_1r_2]\cup[rad_D(A) + Soc(D):_Dr_1]\cup[rad_D(A) + Soc(D):_Dr_2]$ and $r_1r_2\notin[A + Soc(D):_RD]$. Then by our hypothesis $d\notin[0:_Dr_1r_2]$ but $d\in[A:_Dr_1r_2][0:_Dr_1r_2]\cup[rad_D(A) + Soc(D):_Dr_1]\cup[rad_D(A) + Soc(D):_Dr_2]$. Then either $d\in[rad_D(A) + Soc(D):_Dr_1]$ or $d\in[rad_D(A) + Soc(D):_Dr_2]$, that is either $r_1d\in rad_D(A) + Soc(D)$ or $r_2d\in rad_D(A) + Soc(D)$. Therefore, A is a weakly pseudo primary 2-absorbing sub-module of D.

Lemma 12. *Let $A\subsetneq D$ be a sub-module of R-module D. A is a weakly pseudo primary 2-absorbing sub-module of an R-module D if and only if $(0)\neq r_1r_2T\subseteq A$ for $r_1r_2\in R$ and T is sub-module of D, implies either $r_1T\subseteq rad_D$*

$(A) + Soc(D)$ *or* $r_2T \subseteq rad_D(A) + Soc(D)$ *or* $r_1r_2 \in [A + Soc(D):_R D]$.

Proof. Assume that A is weakly pseudo primary 2-absorbing sub-module of an R-module D and $(0) \neq r_1r_2T \subseteq A$ for $r_1r_2 \in R$, T is sub-module of D. Assume that $r_1T \nsubseteq rad_D(A) + Soc(D)$ and $r_2T \nsubseteq rad_D(A) + Soc(D)$ and $r_1r_2 \notin [A + Soc(D):_R D]$. Then there exist t_1, $t_2 \in T$ such that either $r_1t_1 \notin rad_D(A) + Soc(D)$ or $r_2t_2 \notin rad_D(A) + Soc(D)$. Now since $0 \neq r_1r_2t_1 \in A$, $r_1r_2 \notin [A + Soc(D):_R D]$, and A is weakly pseudo primary 2-absorbing sub-module. Then by Proposition 11, $t_1 \in [A:_D r_1r_2] \subseteq [0:_D r_1r_2] \cup [rad_D(A) + Soc(D):_D r_1] \cup [rad_D(A) + Soc(D):_D r_2]$ implies that $t \in [0:_D r_1r_2] \cup [rad_D(A) + Soc(D):_D r_1] \cup [rad_D(A) + Soc(D):_D r_2]$. But $r_1r_2t \neq 0$ and $r_1t_1 \notin rad_D(A) + Soc(D)$, that is $t_1 \notin [0:_D r^2]$ and $t_1 \notin [rad_D(A) + Soc(D):_D r_1]$. Thus $t_1 \in [rad_D(A) + Soc(D):_D r_2]$, that is $r_2t_1 \in rad_D(A) + Soc(D)$. Also since $0 \neq r_1r_2t_2 \in A$ and $r_2t_2 \notin rad_D(A) + Soc(D)$ and $r_1r_2 \notin [A + Soc(D):_R D]$, it follows that $r_1t_2 \subseteq rad_D(A) + Soc(D)$. Now, $0 \neq r_1r_2(t_1 + t_2) \in A$ and $r_1r_2 \notin [A + Soc(D):_R D]$, implies that $t_1 + t_2 \in [A:_D r_1r_2]$ and $t_1t_2 \notin [0:_D r_1r_2]$. It follows by Proposition 11. we have $t_1 + t_2 \in [rad_D(A) + Soc(D):_D r_1] \cup [rad_D(A) + Soc(D):_D r_2]$. That is either $t_1 + t_2 \in [rad_D(A) + Soc(D):_D r_1]$ or $t_1 + t_2 \in [rad_D(A) + Soc(D):_D r_2]$. If $t_1 + t_2 \in [rad_D(A) + Soc(D):_D r_1]$, that is $r_1(t_1 + t_2) = r_1t_1 + r_1t_2 \in rad_D(A) + Soc(D)$. and we have $r_1t_2 \subseteq rad_D(A) + Soc(D)$. That is $r_1t_1 \in rad_D(A) + Soc(D)$. which is a contradiction. If $t_1 + t_2 \in [rad_D(A) + Soc(D):_D r_2]$, that is $r_2(t_1 + t_2) = r_2t_1 + r_2t_2 \in rad_D(A) + Soc(D)$. and we have $r_2t_1 \subseteq rad_D(A) + Soc(D)$. That is $r_2t_2 \in rad_D(A) + Soc(D)$. which is a contradiction. Hence either $r_1T \subseteq rad_D(A) + Soc(D)$ or $r_2T \subseteq rad_D(A) + Soc(D)$ or $r_1r_2 \in [A + Soc(D):_R D]$.
Conversely, Assume that $0 \neq r_1r_2d \in A$, for $r_1, r_2 \in R$, $d \in D$ that is $(0) \neq r_1r_2(d) \subseteq A$, hence by our hypothesis $r_1(d) \subseteq rad_D(A) + Soc(D)$ or $r_2(d) \subseteq rad_D(A) + Soc(D)$ or $r_1r_2 \in [A + Soc(D):_R D]$. Hence, A weakly pseudo primary 2-absorbing sub-module.

Proposition 13. *Let $A \subsetneq D$ be a sub-module of R-module D. A is a weakly pseudo primary 2-absorbing sub-module of a cyclic R-module D if and only if for $r_1r_2 \in R$ with $r_1r_2 \notin [A + Soc(D):_R D]$ we have $[A:_R r_1r_2d] \subseteq [0:_R r_1r_2d] \cup [rad_D(A) + Soc(D):_R r_1d] \cup [rad_D(A) + Soc(D):_R r_2d]$ for some $d \in D$.*

Proof. Assume that A is a weakly pseudo primary 2-absorbing sub-module of a cyclic R-module D and for any $r_1r_2 \in R$ with $r_1r_2 \notin [A + Soc(D):_R D]$, let $x \in [A:_R r_1r_2d]$ to prove $x \in [0:_R r_1r_2d] \cup [rad_D(A) + Soc(D):_R r_1d] \cup [rad_D(A) + Soc(D):_R r_2d]$. Implies that $r_1r_2xd \in A$, that is $r_1r_2(xd) \subseteq A$. If $0 \neq r_1r_2(xd) \subseteq A$ since A is a weakly pseudo primary 2-absorbing sub-module and $r_1r_2 \notin [A + Soc(D):_R D]$, implies that either $r_1(xd) \subseteq rad_D(A) + Soc(D)$ or $r_2(xd) \subseteq rad_D(A) + Soc(D)$ (by Lemma 12) that is either $x \in [rad_D(A) + Soc(D):_R r_1d]$ or $x \in [rad_D(A) + Soc(D):_R r_2d]$. If $0 = r_1r_2(xd) \subseteq A$, implies that $x \in [0:_R r_1r_2d]$. That is $x \in [0:_R r_1r_2d] \cup [rad_D(A) + Soc(D):_R r_1d] \cup [rad_D(A) + Soc(D):_R r_2d]$ Hence $[A:_R r_1r_2d] \subseteq [0:_R r_1r_2d] \cup [rad_D(A) + Soc(D):_R r_1d] \cup [rad_D(A) + Soc(D):_R r_2d]$.
Conversely, Assume that $D = (d_1)$ for some $d_1 \in D$, and $0 \neq r_1r_2d \in A$, for $r_1, r_2 \in R$, $d \in D$ with $r_1r_2 \notin [A + Soc(D):_R D]$. Since $d \in D$ then $d = xd_1$ for some $x \in R$. That is $0 \neq r_1r_2xd_1 \in A$. It follows $x \in [A:_R r_1r_2d_1] \subseteq [0:_R r_1r_2d_1] \cup [rad_D(A) + Soc(D):_R r_1d_1] \cup [rad_D(A) + Soc(D):_R r_2d_1]$. Then $x \in [0:_R r_1r_2d_1] \cup [rad_D(A) + Soc(D):_R r_1d_1] \cup [rad_D(A) + Soc(D):_R r_2d_1]$ but $x \notin [0:_R r_1r_2d_1]$ (sines $0 \neq r_1r_2xd_1$). Then $x \in [rad_D(A) + Soc(D):_R r_1d_1] \cup [rad_D(A) + Soc(D):_R r_2d_1]$. Then either $x \in [rad_D(A) + Soc(D):_R r_1d_1]$ or $x \in [rad_D(A) + Soc(D):_R r_2d_1]$. That is either $r_1xd_1 \in rad_D(A) + Soc(D)$ or $r_2xd_1 \in rad_D(A) + Soc(D)$ Then either $r_1d \in rad_D(A) + Soc(D)$ or $r_2d \in rad_D(A) + Soc(D)$ Therefore A is a weakly pseudo primary 2-absorbing sub-module of a cyclic R-module D.

Theorem 14. *Let $A \subsetneq D$ be a sub-module of R-module D, A is a weakly pseudo primary 2-absorbing sub-module of D if and only if $(0) \neq Q_1Q_2T \subseteq A$ for some ideal Q_1, Q_2 of R and sub-module T of D, implies that either $Q_1T \subseteq rad_D(A) + Soc(D)$ or $Q_2T \subseteq rad_D(A) + Soc(D)$ or $Q_1Q_2 \subseteq [A + Soc(D):_R D]$.*

Proof. Assume that $(0) \neq Q_1Q_2L \subseteq A$ for some ideal Q_1, Q_2 of R and sub-module T of D. With $Q_1Q_2 \nsubseteq [A + Soc(D):_R D]$, to prove $Q_1T \subseteq rad_D(A) + Soc(D)$ or $Q_2T \subseteq rad_D(A) + Soc(D)$. Suppose that $Q_1T \nsubseteq rad_D(A) + Soc(D)$ and $Q_2T \nsubseteq rad_D(A) + Soc(D)$. That is there exist $n_1 \in Q_1$ and $m_1 \in Q_2$, such that $n_1T \nsubseteq rad_D(A) + Soc(D)$ and $m_1T \nsubseteq rad_D(A) + Soc(D)$. Now, $(0) \neq n_1m_1T \subseteq A$. Since A is a weakly pseudo primary 2-absorbing sub-module of D. Then by Lemma 12, either $n_1T \subseteq rad_D(A) + Soc(D)$ or $m_1T \subseteq rad_D(A) + Soc(D)$ or $n_1m_1 \in [A + Soc(D):_R D]$. Since $Q_1Q_2 \nsubseteq [A + Soc(D):_R D]$. Then there exist $n_2 \in Q_1$ and $m_2 \in Q_2$, such that $n_2m_2 \notin [A + Soc(D):_R D]$. But $0 \neq n_2m_2T \in A$ and A is a weakly pseudo primary 2-absorbing sub-module of D with $n_2m_2 \notin [A + Soc(D):_R D]$. Then by Lemma 12, either $n_2T \subseteq rad_D(A) + Soc(D)$ or $m_2T \subseteq rad_D(A) + Soc(D)$. Now, we have three cases.

Case 1. If $n_2T \subseteq rad_D(A) + Soc(D)$ and $m_2T \nsubseteq rad_D(A) + Soc(D)$. Since $(0) \neq n_1m_2T \in A$, $m_2T \nsubseteq rad_D(A) + Soc(D)$ and $n_1T \nsubseteq rad_D(A) + Soc(D)$, so that by Lemma 12, we have $n_1m_2 \in [A + Soc(D):_R D]$. Since $n_2T \subseteq rad_D(A) + Soc(D)$ and $n_1T \nsubseteq rad_D(A) + Soc(D)$. We get $(n_1 + n_2)T \nsubseteq rad_D(A) + Soc(D)$. on the other side $(0) \neq (n_1 + n_2)m_2T \subseteq A$, and A is a weakly pseudo primary 2-absorbing sub-module, $(n_1 + n_2)T \nsubseteq rad_D$

$(A)+Soc(D)$ and $m_2T\nsubseteq rad_D(A)+Soc(D)$. Then by Lemma 12, $(n_1+n_2)m_2=n_1m_2+n_2m_2\in[A+Soc(D):_RD]$. But $n_1m_2\in[A+Soc(D):_RD]$ we have that $n_2m_2\in[A+Soc(D):_RD]$, which is a contradiction.

Case 2. If $m_2T\subseteq rad_D(A)+Soc(D)$ and $n_2T\nsubseteq rad_D(A)+Soc(D)$. Similarly, to case 1, we get a contradiction.

Case 3. If $n_2T\subseteq rad_D(A)+Soc(D)$ and $m_2T\subseteq rad_D(A)+Soc(D)$, since $m_2T\subseteq rad_D(A)+Soc(D)$ and $m_1T\nsubseteq rad_D(A)+Soc(D)$, we get $(m_1+m_2)T\nsubseteq rad_D(A)+Soc(D)$. But $(0)\neq n_1(m_1+m_2)T\subseteq A$ and A is a weakly pseudo primary 2-absorbing sub-module with $n_1T\nsubseteq rad_D(A)+Soc(D)$ and $(m_1+m_2)T\nsubseteq rad_D(A)+Soc(D)$. Then by Lemma 12, we get $n_1(m_1+m_2)\in[A+Soc(D):_RD]$. Since $n_1m_1\in[A+Soc(D):_RD]$ and $n_1m_1+n_1m_2\in[A+Soc(D):_RD]$, implies that $n_1m_2\in[A+Soc(D):_RD]$. Since $(0)\neq(n_1+n_2)m_1T\in A$ and $m_1T\nsubseteq rad_D(A)+Soc(D)$ and $(n_1+n_2)T\nsubseteq rad_D(A)+Soc(D)$ and A is a weakly pseudo primary 2-absorbing sub-module. Then by Lemma 12, we get $(n_1+n_2)m_1\in[A+Soc(D):_RD]$. But $n_1m_1+n_1m_1\in[A+Soc(D):_RD]$, and since $n_1m_1\in[A+Soc(D):_RD]$, we get $n_2m_1\in[A+Soc(D):_RD]$. Since $(0)\neq(n_1+n_2)(m_1+m_2)T\subseteq A$ and $(n_1+n_2)T\nsubseteq[A+Soc(D):_RD]$ and $(m_1+m_2)T\nsubseteq[A+Soc(D):_RD]$. Then by Lemma 12, we get $(n_1+n_2)(m_1+m_2)=n_1m_1+n_1m_2+n_2m_1+n_2m_2\in[A+Soc(D):_RD]$. Since $n_1m_1\in[A+Soc(D):_RD]$, $n_2m_1\in[A+Soc(D):_RD]$ and $n_1m_2\in[A+Soc(D):_RD]$, we get $n_2m_2\in[A+Soc(D):_RD]$, which is a contradiction. Consequently either $Q_1T\subseteq rad_D(A)+Soc(D)$ or $Q_2T\subseteq rad_D(A)+Soc(D)$.

Conversely, assume that $0\neq r_1r_2d\subseteq A$ for $r_1,r_2\in R$ and $d\in D$, that is $0\neq(r_1)(r_2)T\subseteq A$. Then by hypothesis either $(r_1)T\subseteq rad_D(A)+Soc(D)$ or $(r_2)T\subseteq rad_D(A)+Soc(D)$ or $(r_1)(r_2)\subseteq[A+Soc(D):_RD]$. That is either $r_1T\subseteq rad_D(A)+Soc(D)$ or $r_2T\subseteq rad_D(A)+Soc(D)$ or $r_1r_2\subseteq[A+Soc(D):_RD]$. Then by Lemma 12, we get A is a weakly pseudo primary 2-absorbing sub-module of a R-module D.

The following corollaries are direct consequences of a *Theorem 14*.

Corollary 16. *Let $A\subsetneq D$ be a sub-module of R-module D, A is a weakly pseudo primary 2-absorbing sub-module of an R-module D if and only if $(0)\neq Q_1Q_2d\subseteq A$ for some ideals $Q_1.Q_2$ of R and $d\in D$, implies that $Q_1d\subseteq rad_D(A)+Soc(D)$ or $Q_2d\subseteq rad_D(A)+Soc(D)$ or $Q_1Q_2\subseteq[A+Soc(D):_RD]$.*

Corollary 17. *Let $A\subsetneq D$ be a sub-module of R-module D, A is a weakly pseudo primary 2-absorbing sub-module of an R-module D if and only if $(0)\neq rQT\subseteq A$, for some $r\in R$ and for some ideal Q of R and sub-module T of D, implies that $rT\subseteq rad_D(A)+Soc(D)$ or $QT\subseteq rad_D(A)+Soc(D)$ or $rQ\subseteq[A+Soc(D):_RD]$.*

Corollary 18. *Let $A\subsetneq D$ be a sub-module of R-module D. A is a weakly pseudo primary 2-absorbing sub-module of an R-module D if and only if $(0)\neq rQd\subseteq A$, for some $r\in R$ and for some ideal Q of R and $d\in D$, implies that $rQ\subseteq rad_D(A)+Soc(D)$ or $Qd\subseteq rad_D(A)+Soc(D)$ or $rQ\subseteq[A+Soc(D):_RD]$.*

Remark 19. [11] Suppose that R-module D is semi-simple if and only if for any sub-module A of D, $Soc\left(\frac{D}{A}\right)=\frac{Soc(D)+A}{A}$.

Proposition 20. *Let $A\subsetneq D$ be a sub-module of R-module D. A is a weakly pseudo primary 2-absorbing sub-module of D with D is semi-simple, and T is sub-module of D with $T\subseteq A$, then $\frac{A}{T}$ is weakly pseudo primary 2-absorbing sub-module of an R-module $\frac{D}{T}$.*

Proof. Assume that A is a weakly pseudo primary 2-absorbing sub-module of an R- module D and T is a sub-module of D and let $T\subseteq A$. let $0\neq r_1r_2(d+T)=r_1r_2d+T\in\frac{A}{T}$ with $r_1,r_2\in R$ and $d+T\in\frac{D}{T}$, $d\in D$, implies that $r_1r_2d\in A$ If $r_1r_2d=0$ then $r_1r_2(d+T)=0$ which is contradiction. Thus $0\neq r_1r_2d\in A$ and since A is a weakly pseudo primary 2-absorbing sub-module then either $r_1d\in rad_D(A)+Soc(D)=rad_D(A)+T+Soc(D)$ or $r_2d\in rad_D(A)+Soc(D)=rad_D(A)+T+Soc(D)$ or $r_1r_2D\subseteq A+Soc(D)$. Since ($T\subseteq A\subseteq rad_D(A)$ then $rad_D(A)+T=rad_D(A)$). It follows that is either $r_1(d+T)\in\frac{rad_D(A)+T+Soc(D)}{T}$ or $r_2(d+T)\in\frac{rad_D(A)+T+Soc(D)}{T}$ or $r_1r_2\frac{D}{T}\subseteq\frac{A+T+Soc(D)}{T}$. That is either $r_1(d+T)\in\frac{rad_D(A)}{T}+\frac{T+Soc(D)}{T}$ or $r_2(d+T)\in\frac{rad_D(A)}{T}+\frac{T+Soc(D)}{T}$ or $r_1r_2\frac{D}{T}\subseteq\frac{A}{T}+\frac{A+Soc(D)}{T}$. since D is semi-simple then $Soc\left(\frac{D}{T}\right)=\frac{T+Soc(D)}{T}$ by Remark 19 that is either $r_1(d+T)\subseteq rad_{\frac{D}{T}}\frac{A}{T}+Soc\left(\frac{D}{T}\right)$ or $r_2(d+T)\subseteq rad_{\frac{D}{T}}\frac{A}{T}+Soc\left(\frac{D}{T}\right)$ or $r_1r_2\in\left[\frac{A}{T}+Soc\left(\frac{D}{T}\right):_R\frac{D}{T}\right]$ Hence $\frac{A}{T}$ is weakly pseudo primary 2-absorbing sub-module of an R-module $\frac{D}{T}$.

Proposition 21. *Let D be a semi-simple R- module, and A,B are sub-modules of R- module D with $B\subseteq A$ and $rad_D(B)\subseteq rad_D(A)$. If B and $\frac{A}{B}$ are weakly pseudo primary 2-absorbing sub-modules of D, $\frac{D}{B}$ respectively, then A is weakly pseudo primary 2-absorbing sub-module.*

Proof. Assume that $0\neq r_1r_2d\in A$ for $r_1,r_2\in R$ and $d\in D$, then $0\neq r_1r_2(d+B)\in\frac{A}{B}$ If $0\neq r_1r_2d\in B$ and B is weakly pseudo primary 2-absorbing sub-module, implies that either $r_1d\in rad_D(B)+Soc(D)\subseteq rad_D(A)+Soc(D)$ or $r_2d\in rad_D(B)+Soc(D)\subseteq rad_D(A)+Soc(D)$ or $r_1r_2D\subseteq B+Soc(D)\subseteq A+Soc(D)$, thus A is a weakly pseudo primary 2-absorbing sub-module. Assume that $r_1r_2d\notin B$, it follows that $0\neq r_1r_2(d+B)\in \frac{A}{B}$, since $\frac{A}{B}$ is weakly pseudo primary 2-absorbing sub-module of $\frac{D}{B}$, implies that either $r_1(d+B)\subseteq rad_{\frac{D}{B}}\left(\frac{A}{B}\right)+Soc\left(\frac{D}{B}\right)$ or $r_2(d+B)\subseteq rad_{\frac{D}{B}}\left(\frac{A}{B}\right)+Soc\left(\frac{D}{B}\right)$ or $r_1r_2\frac{D}{B}\subseteq\frac{A}{B}+Soc\left(\frac{D}{B}\right)$, since D is semi-simple then $Soc\left(\frac{D}{B}\right)=\frac{B+Soc(D)}{B}$ by Remark 19 it follows that either $r_1(d+B)\subseteq\frac{rad_D(A)}{B}+\frac{B+Soc(D)}{B}$ or $r_2(d+B)\subseteq\frac{rad_D(A)}{B}+\frac{B+Soc(D)}{B}$ or $r_1r_2\frac{D}{B}\subseteq\frac{A}{B}+\frac{B+Soc(D)}{B}$, since $B\subseteq A$ implies that $B+Soc(D)\subseteq A+Soc(D)$, hence either $r_1(d+B)\subseteq\frac{rad_D(A)}{B}+\frac{A+Soc(D)}{B}=\frac{rad_D(A)+Soc(D)}{B}$ or $r_2(d+B)\subseteq\frac{rad_D(A)}{B}+\frac{A+Soc(D)}{B}=\frac{rad_D(A)+Soc(D)}{B}$ or $r_1r_2\frac{D}{B}\subseteq\frac{A}{B}+\frac{A+Soc(D)}{B}$, implies that either $r_1(d+B)\subseteq\frac{rad_D(A)+Soc(D)}{B}$ or $r_2(d+B)\subseteq\frac{rad_D(A)+Soc(D)}{B}$ or $r_1r_2\frac{D}{B}\subseteq\frac{A+Soc(D)}{B}$, it follows either $r_1d\subseteq rad_D(A)+Soc(D)$ or $r_2d\subseteq rad_D(A)+Soc(D)$ or $r_1r_2\in[A+Soc(D):_R D]$ Therefore, A is a weakly pseudo primary 2-absorbing sub-module.

Lemma 22. [7] *Suppose that D be an R-module, T is essential sub-module of D, Then $Soc(D)=Soc(T)$.*

Proposition 23. *Suppose that A and T are sub-modules of R-modules D, with A is a proper sub-module of T, $rad_D(T)\subseteq rad_T(A)$, and T an essential sub-module of D. If T is a weakly pseudo primary 2-absorbing sub-module of D, then A is a weakly pseudo primary 2-absorbing sub-module of T.*

Proof. Assume that $(0)\neq r_1r_2L\in A$ for $r_1,r_2\in R$, L is sub-module of T, that is L is a sub-module of D and $(0)\neq r_1r_2L\in T$. Since T is weakly pseudo primary 2-absorbing sub-module of D, it follows that by *Lemma 16* either $r_1L\subseteq rad_D(T)+Soc(D)$ or $r_2L\subseteq rad_D(T)+Soc(D)$ or $r_1r_2\in[T+Soc(D):_R D]$. But T is essential sub-module of D, then by Proposition *Lemma 26* $Soc(D)=Soc(T)$, and since $rad_D(T)\subseteq rad_T(A)$ Thus either $r_1L\subseteq rad_T(A)+Soc(T)$ or $r_2L\subseteq rad_T(A)+Soc(T)$ or $r_1r_2\in[A+Soc(T):_R D]\subseteq[A+Soc(T):_R T]$. Hence, A is a weakly pseudo primary 2-absorbing sub-module of T.

Lemma 24. [7] *[Module Law] Suppose that A, B and C are a sub-module of an $R-$module D, with $B\subseteq C$. Then $(A+B)\cap C=(A\cap C)+B=(A\cap C)+(B\cap C)$.*

Lemma 25. [7] *Suppose that A is a sub-module of an $R-$module D, then $Soc(A)=A\cap Soc(D)$.*

Proposition 26. *Assume that A and B are sub-modules of R-module D, with B is no contained in A and $Soc(D)\subseteq B$. If A is a weakly pseudo primary 2-absorbing sub-module of D, Then $A\cap B$ is a weakly pseudo primary 2-absorbing sub-module of B.*

Proof. Since B is no contained in A, then $A\cap B$ is a proper sub-module of B. Let $(o)\neq r_1r_2L\subseteq A\cap B$ for some $r_1,r_2\in R$ and L is a sub-module of B, that is L is a sub-module of D. It follows that $o\neq r_1r_2L\subseteq A$, But A is a weakly pseudo primary 2-absorbing sub-module, then by Lemma 12 either $r_1L\subseteq rad_D(A)+Soc(D)$ or $r_2L\subseteq rad_D(A)+Soc(D)$ or $r_1r_2D\subseteq A+Soc(D)$. so then either $r_1L\subseteq(rad_D(A)+Soc(D))\cap B$ or $r_2L\subseteq(rad_D(A)+Soc(D))\cap B$ or $r_1r_2D\subseteq(A+Soc(D))\cap B$. since $Soc(D)\subseteq B$, then by module law (Lemma 24). either $r_1L\subseteq(rad_D(A)\cap B)+(B\cap Soc(D))$ or $r_2L\subseteq(rad_D(A)\cap B)+(B\cap Soc(D))$ or $r_1r_2D\subseteq(A\cap B)+(B\cap Soc(D))$. Thus by Lemma 25 $B\cap Soc(D)=Soc(B)$. it follows that either $r_1L\subseteq(rad_D(A)\cap B)+Soc(B)$ or $r_2L\subseteq(rad_D(A)\cap B)+Soc(B)$ or $r^2D\subseteq(A\cap B)+Soc(B)$. Since $B\subseteq rad_D(B)$ and $rad_D(A)\cap rad_D(B)=rad_D(A\cap B)$. Then either $r_1Q\subseteq rad_D(A\cap B)+Soc(B)$ or $r_2Q\subseteq rad_D(A\cap B)+Soc(B)$ or $r^2D\subseteq(A\cap B)+Soc(B)$. Hence, $A\cap B$ is a weakly pseudo primary 2-absorbing sub-module of B.
Under certain conditions, the intersection of each pair of distinct weakly pseudo primary 2-absorbing sub-module is a weakly pseudo primary 2-absorbing sub-module in general.

Proposition 27. *Assume that A and B are weakly pseudo primary 2-absorbing sub-modules of R-module D, with B is no contained in A and either $Soc(D)\subseteq A$ or $Soc(D)\subseteq B$. Then $A\cap B$ is a weakly pseudo primary 2-absorbing sub-module of D.*

Proof. Since B is not contained in A and B is a proper sub-module of D, it implies that $A\cap B$ is a proper sub-module of D. Assume that $Soc(D)\subseteq A$ But $Soc(D)\nsubseteq B$. Let $o\neq Q_1Q_2L\subseteq A\cap B$ for some Q_1,Q_2 be an ideal in R and L is a sub-module of B, that is, L is a sub-module of D. It follows that $o\neq Q_1Q_2L\subseteq A$ and $o\neq Q_1Q_2L\subseteq B$, But A and B are a weakly pseudo primary 2-absorbing sub-module, then by Theorem 14 either $Q_1L\subseteq rad_D(A)+Soc(D)$ or $Q_2L\subseteq rad_D(A)+Soc(D)$ or $Q_1Q_2D\subseteq A+Soc(D)$ and either $Q_1L\subseteq rad_D(B)+Soc(D)$ or $Q_2L\subseteq rad_D(B)+Soc(D)$ or $Q_1Q_2\,D\subseteq B+Soc(D)$. it follows that either $Q_1L\subseteq(rad_D(A)+Soc(D))\cap(rad_D(B)+Soc(D))$ or $Q_2L\subseteq(rad_D(A)+Soc(D))\cap(rad_D(B)+Soc(D))$ or $Q_1Q_2D\subseteq(A+Soc(D))\cap(B+Sco(D))$. since $Soc(D)\subseteq B$. Then, by module law (Lemma 24). either $Q_1L\subseteq(rad_D(A)\cap rad_D(B))+Soc(D)$ or $Q_2L\subseteq(rad_D(A)\cap rad_D(B))+Soc(D)$ or $Q_1Q_2D\subseteq(A\cap B)+Soc(D)$. That is either $Q_1L\subseteq rad_D(A\cap B)+Soc(D)$ or $Q_2L\subseteq rad_D(A\cap B)+Soc(D)$ or $Q_1Q_2D\subseteq(A\cap B)+Soc(D)$. By Theorem 14 Hence, $A\cap B$ is a weakly pseudo primary 2-absorbing sub-module of D. similar if $Soc(D)\subseteq B$.

Lemma 28. [7] *Suppose that M and N be $R-$module and $f:M\rightarrow N$ be an R-epimorphism, then $f(Soc(M))\subseteq Soc(N)$.*

Lemma 29. [7] *Suppose that $f: D_1 \to D_2$ be an R-epimorphism, with $kef(f) \subseteq A$ then $f(rad_{D_1}(A)) \subseteq rad_{D_2}(f(A))$.*

Proposition 30. *Suppose that $f: D_1 \to D_2$ be an R-epimorphism and A is a weakly pseudo primary 2-absorbing sub-module of D_1, with $ker(f) \subseteq A$ then $f(A)$ is a weakly pseudo primary 2-absorbing sub-module of D_2.*

Proof. It's clear that $f(A)$ is a sub-module of D_2. Let $o \neq r_1 r_2 d_2 \in f(A)$ for some $r_1, r_2 \in R$ and $d_2 \in D_2$, and let $r_1 r_2 \notin [f(A) + Soc(D_2) :_R D_2]$ to prove either $r_1 d_2 \in rad_{D_2}(f(A)) + Soc(D_2)$ or $r_2 d_2 \in rad_{D_2}(f(A)) + Soc(D_2)$. Since f is onto then $o \neq r_1 r_2 f(d_1) \in f(A)$ for some $d_1 \in D_1$ that is $f(r_1 r_2 d_1) = f(q)$ for some $q \in A$. That is $r_1 r_2 d_1 - q \in ker(f) \subseteq A$. implies that $r_1 r_2 d_1 \in A$, that is $d_1 \in [A :_{D_1} r_1 r_2]$, it follows by Proposition 11 then $d_1 \in [A :_{D_1} r_1 r_2] \subseteq [0 : D_1 r_1 r_2] \cup [rad_{D_1}(A) + Soc(D_1) :_{D_1} r_1] \cup [rad_{D_1}(A) + Soc(D_1) :_{D_1} r_2]$. But A is a weakly pseudo primary 2-absorbing sub-module of D_1 so $r_1 r_2 d_1 \neq 0$ that is $d_1 \notin [0 :_{D_1} r_1 r_2]$, therefore either $d_1 \in [rad_{D_1}(A) + Soc(D_1) :_{D_1} r_1]$, or $d_1 \in [rad_{D_1}(A) + Soc(D_1) :_{D_1} r_2]$. That is either $r_1 d_1 \in rad_{D_1}(A) + Soc(D_1)$ or $r_2 d_1 \in rad_{D_1}(A) + Soc(D_1)$. So either $r_1 f(d_1) \in f(rad_{D_1}(A)) + f(Soc(D_1)) \subseteq rad_{D_2}(f(A)) + Soc(D_2)$ or $r_2 f(d_1) \in f(rad_{D_1}(A)) + f(Soc(D_1)) \subseteq rad_{D_2}(f(A)) + Soc(D_2)$ (by Lemma 28 $f(Soc(D_1)) \subseteq Soc(D_2)$ and Lemma 29 $f(rad_{D_1}(A)) \subseteq rad_{D_2}(f(A))$). Thus either $r_1 d_2 \in rad_{D_2}(f(A)) + Soc(D_2)$. $r_2 d_2 \in rad_{D_2}(f(A)) + Soc(D_2)$. Therefore, $f(A)$ is a weakly pseudo primary 2-absorbing sub-module of D_2.

Proposition 31. *Suppose that $f: D_1 \to D_2$ be an R-epimorphism and A is a weakly pseudo primary 2-absorbing sub-module of D_2, then $f^{-1}(A)$ is a weakly pseudo primary 2-absorbing sub-module of D_1.*

Proof. Its clear that $f^{-1}(A)$ is sub-module of D_1. Let $o \neq r_1 r_2 d \in f^{-1}(A)$ for some $r_1, r_2 \in R$ and $d \in D_1$, then $0 \neq r_1 r_2 f(d) \in A$ since A is a weakly pseudo primary 2-absorbing sub-module of D_2, implies that either $r_1 f(d) \in rad_{D_2}(A) + Soc(D_2)$ or $r_2 f(d) \in rad_{D_2}(A) + Soc(D_2)$ or $r_1 r_2 D_2 \subseteq A + Soc(D_2)$. Since f is R-epimorphism then $D_2 = f(D)$. It follows that either $r_1 d \in f^{-1} rad_{D_2}(A) + f^{-1}(Soc(D_2)) \subseteq rad_{D_1}(f^{-1}(A)) + Soc(D)$ or $r_2 d \in f^{-1} rad_{D_2}(A) + f^{-1}(Soc(D_2)) \subseteq rad_{D_1}(f^{-1}(A)) + Soc(D)$ or $r_1 r_2 D \subseteq f^{-1}(A) + Soc(D)$. (by Lemma 28 and Lemma 29). Hence $f^{-1}(A)$ is a weakly pseudo primary 2-absorbing sub-module of D_1.

Lemma 32. [8] *Let $V = \bigoplus_{i \in \wedge} V_i$ be an R-module and V_i is an R-module for each $i \in \wedge$, then $Soc(V) = \bigoplus_{i \in \wedge} Sco(V_i)$.*

Proposition 33. *Suppose that $D = D_1 \oplus D_2$ be an R-module with D_1, D_2 be R-module and $A = A_1 \oplus A_2$ be sub-module of D, where A_1, A_2 sub-modules of D_1, D_2 respectively and $rad_D(A) \subseteq Soc(D)$. If A is a weakly pseudo primary 2-absorbing sub-module of D, then A_1 and A_2 are weakly pseudo primary 2-absorbing sub-modules of D_1 and D_2 respectively.*

Proof. Assume that A is a weakly pseudo primary 2-absorbing sub-module of D and let $o \neq r_1 r_2 d_1 \in A_1$ for some $r_1 r_2 \in R$ and $d_1 \in D_1$, it follows that $(0,0) \neq r_1 r_2 (d_1, 0) \in A_1 \oplus A_2$, but $A = A_1 \oplus A_2$ is a weakly pseudo primary 2-absorbing sub-module of D, then either $r_1(d_1, 0) \in rad_D(A_1 \oplus A_2) + Soc(D_1 \oplus D_2)$ or $r_2(d_1, 0) \in rad_D(A_1 \oplus A_2) + Soc(D_1 \oplus D_2)$ or $r_1 r_2 D \subseteq (A_1 \oplus A_2) + Soc(D_1 \oplus D_2)$. But $rad_D(A) \subseteq Soc(D)$, then $rad_D(A) + Soc(D) = Soc(D) = Soc(D_1 \oplus D_2) = Soc(D_1) \oplus Soc(D_2)$. (by Lemma 32). Thus either $r_1(d_1, 0) \in Soc(D_1) \oplus Soc(D_2)$ or $r_2(d_1, 0) \in Soc(D_1) \oplus Soc(D_2)$ or $r_1 r_2 (D_1 \oplus D_2) \subseteq Soc(D_1) \oplus Soc(D_2)$, it follows that either $rd_1 \in Soc(D_1) \subseteq rad_{D_1}(A_1) + Soc(D_1)$ or $rd_2 \in Soc(D_1) \subseteq rad_{D_1}(A_1) + Soc(D_1)$ or $r_1 r_2 D \subseteq Soc(D_1) \subseteq A_1 + Soc(D_1)$. Hence, A_1 is a weakly pseudo primary 2-absorbing sub-module of D_1. In the same way A_2 is a weakly pseudo primary 2-absorbing sub-module of D_2.

Proposition 34. *Suppose that $D = D_1 \oplus D_2$ be an R-module with D_1, D_2 be R-module and A is a proper sub-module of D_1, with $rad_{D_1}(A) \subseteq Soc(D_1)$, and $Soc(D_2) = D_2$. if A is a weakly pseudo primary 2-absorbing sub-module of D_1 then $A \oplus D_2$ is a weakly pseudo primary 2-absorbing sub-module of D.*

Proof. Assume that $(0,0) \neq r_1 r_2 (d_1, d_2) \in A \oplus D_2$ for $r_1, r_2 \in R$ $(d_1, d_2) \in D_1 \oplus D_2$ where $0 \neq d_1 \in D_1$, $0 \neq d_2 \in D_2$, implies that $0 \neq r_1 r_2 d_1 \in A$ but A is a weakly pseudo primary 2-absorbing sub-module of D_1, implies that either $r_1 d_1 \in rad_{D_1}(A) + Soc(D_1)$ or $r_2 d_1 \in rad_{D_1}(A) + Soc(D_1)$ or $r_1 r_2 D_1 \subseteq A + Soc(D_1)$. Since $rad_{D_1}(A) \subseteq Soc(D_1)$, then $rad_{D_1}(A) + Soc(D_1) = Soc(D_1)$, it follows that either $r_1 d_1 \in Soc(D_1)$ or $r_2 d_1 \in Soc(D_1)$ or $r_1 r_2 D_1 \subseteq Soc(D_1)$. But $Soc(D_2) = D_2$, so we have either $r_1(d_1, d_2) \in Soc(D_1) \oplus Soc(D_2) = Soc(D_1 \oplus D_2) \subseteq rad_D(A \oplus D_2) + Soc(D)$ or $r_2(d_1, d_2) \in Soc(D_1) \oplus Soc(D_2) = Soc(D_1 \oplus D_2) \subseteq rad_D(A \oplus D_2) + Soc(D)$ or $r_1 r_2 (D_1 \oplus D_2) \subseteq Soc(D_1) \oplus Soc(D_2) = Soc(D) \subseteq A \oplus D_2 + Soc(D)$. (by *Lemma 34*) Hence, $A \oplus D_2$ is a weakly pseudo primary 2-absorbing sub-module of D.

Proposition 35. *Let $A \subsetneq D$ is a sub-module of R- module D, with $rad_D(A)$ is 2-absorbing sub-module of D. Then A is a weakly pseudo primary 2-absorbing sub-module of D.*

Proof. Assume that $0 \neq rQd \in A$ for $r \in R$, Q be an ideal of R and $d \in D$ with $rQD \nsubseteq A + Soc(D)$. Then $rQd \in rad_D(A)$, since $rad_D(A)$ is 2-absorbing sub-module of D, by Corollary 18 that is either $rd \in rad_D(A) \subseteq rad_D(A) + Soc(D)$ or $Qd \subseteq rad_D(A) \subseteq rad_D(A) + Soc(D)$ or $rQD \subseteq A \subseteq A + Soc(D)$ then either $rd \in$

$rad_D(A) + Soc(D)$ or $Qd \subseteq rad_D(A) + Soc(D)$ or $rQD \subseteq A + Soc(D)$ Hence, by Corollary 18, A is a weakly pseudo primary 2-absorbing sub-module of D.

Proposition 36. *Let D be an R- module, with* $Soc(A)$ *is a 2-absorbing sub-module of D. If* $A \subsetneq D$ *is a sub-module of D such that* $A \subseteq Sco(D)$*, Then A is a weakly pseudo primary 2-absorbing sub-module of D.*

Proof. Assume that $0 \neq rQT \subseteq A$ for $r \in R$, Q be an ideal of R, and T is sub-module of D, since $A \subseteq Sco(D)$ Then $rQT \subseteq Soc(D)$, since $Soc(D)$ is 2-absorbing sub-module of D, and by Corollary 17 that is either $rT \in Soc(D) \subseteq rad_D(A) + Soc(D)$ or $QT \in Soc(D) \subseteq rad_D(A) +\ Soc(D)$ or $rQD \subseteq Soc(D) = A + Soc(D)$. then either $rT \subseteq rad_D(A) + Soc(D)$ or $QT \subseteq rad_D(A) + Soc(D)$ or $r_1 r_2 D \subseteq A +\ Soc(D)$. Hence, by Corollary 17, A is a weakly pseudo primary 2-absorbing sub-module of D.

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