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ORIGINAL STUDY

$\bigoplus$-Supplemented Semimodules

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Abstract

In this paper, $\bigoplus$-supplemented semimodules are defined as generalizations of $\bigoplus$-supplemented modules. Let S be a semiring. An S-semimodule A is named a $\bigoplus$-supplemented semimodule, if every subsemimodule of A has a supplement which is a direct summand of A. In this paper, we investigate some properties of $\bigoplus$-supplemented semimodules besides generalize certain results on $\bigoplus$-supplemented modules to semimodules.

Keywords: Supplemented semimodules, (Completely) $\bigoplus$-supplemented semimodules, Semiperfect semimodules

1. Introduction

Firstly, let us point that, S will indicate an associative semiring with identity besides A will indicate an unitary left S-semimodule throughout this article. A (left) S-semimodule A is a commutative additive semigroup which has a zero element 0_A, together with a mapping from $S \times A$ into A (sending (s, a) to sa) such that $(r + s)a = ra + sa$, $r(a + b) = ra + rb$, $r(sa) = (rs)a$ besides $0a = r0_A = 0$ for all $a, b \in A$ besides $r, s \in S$. Let N be a subset of A. One say that N is an S-subsemimodule of A, precisely when N is itself an S-semimodule with respect to the operations for A. A subsemimodule N of A is a direct summand of A iff there is a subsemimodule N' of A satisfying $A = N \bigoplus N'$, in particular, any element a of A can be written in a unique manner as $a + a'$, where $a \in N$ and $a' \in N'$ [7, p. 184]. Too to these, for a subsemimodule X of A besides for a direct summand X of A, the notations $X \leq A$ besides $X \leq_{\bigoplus} A$ will be used respectively. $L \leq A$ is named essential in A, indicated by $L \leq_e A$, if $L \cap N \neq 0$ for all non-zero subsemimodule $N \leq A$.

A subsemimodule $N \leq A$ is named small in A (one writes $N \ll A$), if for every subsemimodule $X \leq A$, with $N + X = A$ implies that $X = A$ [14]. The radical of A, symbolized by $Rad(A)$, is the sum of all small subsemimodules of A [14]. A is named hollow, if each proper subsemimodule of A is small in A. A is named local, if it has a single maximal subsemimodule, i.e., a proper subsemimodule which contains all other subsemimodules. A is said to be simple, if it has no nontrivial subsemimodule, besides A is said to be semisimple if it is a direct sum of its simple subsemimodules [3]. The socle of A, symbolized by $Soc(A)$, is the sum of all simple subsemimodules of A [3]. Let L, $K \leq A$. K is named a supplement of L in A if it is minimal with respect to $A = L + K$. A subsemimodule K of A is a supplement (weak supplement) of L in A iff $A = L + K$ besides $L \cap K \ll K$ ($L \cap K \ll A$) [3]. A is supplemented (weakly supplemented) if each subsemimodule L of A has a supplement (weak supplement) in A. Openly, supplemente semimodules are weakly supplemente. $L \leq A$ has ample supplements in A if each subsemimodule K of A such that $A = L + K$ contains a supplement of L in A. A semimodule A is named amply supplemented if every subsemimodule of A has ample supplements in A. Hollow semimodules are ample supplemented. A semimodule A is named lifting (or D_1) if, for all $N \leq A$, there is a decomposition $A = X \bigoplus Y$ such that $X \leq N$ and $N \cap Y$ is small in A [12]. A subsemimodule N of $\leq A$ is named a subtractive subsemimodule of A if $a, a + b \in N$ then $b \in N$ for all $a, b \in A$ ([4,7]). If every subsemimodule of A is subtractive subsemimodule, at that time A is named subtractive. If C is a subtractive subsemimodule, at that time $\frac{A}{C}$ is an R-semimodule [7, p. 165].

In this paper, we introduce $\bigoplus$-supplemented semimodules and investigate their possessions. New characterizations of semiperfect semimodules

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are obtained using $\bigoplus$-supplemente semimodules. In Section 2, we define $\bigoplus$-supplemented semimodules. Furthermore, for any semiring S, we show that any finite direct sum of $\bigoplus$-supplemented S-semimodules is $\bigoplus$-supplemented. In Section 3, we define completely $\bigoplus$-supplemented semimodules. We also show that any $\bigoplus$-supplemented semimodule has D_3 property is completely $\bigoplus$-supplemented.

In what follows, by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_n$ and $\mathbb{Z}/n\mathbb{Z}$ we indicate, respectively, natural numbers, non-negative integers, integers, rational numbers, the semiring of integers modulo n besides the $\mathbb{Z}$-semimodule of integers modulo n.

2. $\bigoplus$-Supplemented Semimodules

In this part, we introduce $\bigoplus$-supplemente semimodules. Mohamed and Müller [10] call a module A $\bigoplus$-supplementd if each submodule N of A has a supplement that is a direct summand of A. Openly, each $\bigoplus$-supplementd module is supplementd, nonetheless a supplementd modul need not be $\bigoplus$-supplemente in general (see [10, Lem. A.4 (2)]). Alike to [10] we have the next definition of $\bigoplus$-supplemented semimodules.

Definition 2.1. An S-semimodule A is named $\bigoplus$-supplemented if for every subsemimodule N of A there is a direct summand K of A such that $A = N + K$ and $N \cap K$ is small in K.

Remark 2.2. Obviously $\bigoplus$-supplemented semimodules are supplemented. In addition, Hollow semimodules and lifting semimodules are $\bigoplus$-supplemented.

Definition 2.3. [2] A semimodule A is named principally $\bigoplus$-supplemented if for each $a \in A$ there exists a direct summand B of A such that $A = Sa + B$ and $Sa \cap B$ is small in B.

Definition 2.4. [2] A semimodule A is named a weak principally $\bigoplus$-supplemented if for each $a \in A$ there exists a direct summand B such that $A = Sa + B$ and $Sa \cap B \ll A$.

Each $\bigoplus$-supplemented semimodule is supplemented. All $\bigoplus$-supplemented semimodules are principally $\bigoplus$-supplemented.

Definition 2.5. [14] A homomorphism $f : A \to B$ of left S-semimodules is named k-quasiregular if whenever $K \leq A$, $a \in A \setminus K$, $a' \in K$, and $f(a) = f(a')$ there exists $s \in \mathrm{Ker}(f)$ such that $a = a' + s$.

Definition 2.6. [14] Let A be a left S-semimodule. A left S-semimodule P together with an S-homomorphism $f : P \to A$ is named a projective cover of A if:

(1) P is projective,
(2) f is small, epimorphism besides k-quasiregular.

By [13], a semiring is named perfect (or semiperfect) if every S-semimodule (or every simple S-semimodule) has a projective cover. Too, a semiring is named semiperfect if each finitely generated S-semimodule has a projective cover. Now alike to [13] the next definition are given.

Definition 2.7. A semimodule A is named semiperfect if each factor semimodule of A has a projective cover.

Mohamed and Müller [10, Coro. 4.43] call a projective module A is semiperfect, iff A is discrete (if A has the conditions (D_1) and (D_2)), iff every submodule of A has a supplement.

Let A be a semimodule. Similar to [10], we consider the next conditions in semimodule theory.

(D_1) For each subsemimodule N of A, A has a decomposition with $A = A_1 \bigoplus A_2$, $A_1 \leq N$ and $A_2 \cap N \ll A_2$.
(D_2) If N is a subsemimodule of A is such that $\frac{A}{N}$ is isomorphic to a summand of A, then N is a summand of A.
(D_3) If A_1 besides A_2 are direct summands of A with $A = A_1 + A_2$, then $A_1 \cap A_2$ is besides a direct summand of A.

Similar to [10], we call a projective subtractive semimodule A is semiperfect, if and only if A is discrete (if A has the conditions (D_1) and (D_2)), if and only if each subsemimodule of A has a supplement. Now, alike to [8, Lemma 1.2], we give the next lemma.

Lemma 2.8. Assume A is a projective subtractive semimodule. Now the next statements are equivalent.

(1) A is semiperfect.
(2) A is supplemented.
(3) A is $\bigoplus$-supplemented.

Proof: (1) ⇔ (2) Using [10, Coro. 4.43]. (1) ⇔ (3) as in the proof of [5], Propo. 1.4]. □

Let A be a semimodule. Similar to [10, Proposition 4.8], A has (D_1) iff A is amply supplementd besides each supplement subsemimodule of A is a direct summand. As a result, every (D_1)-semimodule is $\bigoplus$-supplemented.

Lemma 2.9. Suppose that N and L are sub-semimodules of A with $N+L$ has a supplement H in A besides $N\cap(H+L)$ has a supplement G in N. At that time $H+G$ is a supplement of L in A.

Proof: Lease H be a supplement of $N+L$ in A besides, G be a supplement of $N\cap(H+L)$ in N. Now $A=(N+L)+H$ such that $(N+L)\cap H\ll H$ and $N=[N\cap(H+L)]+G$ such that $(H+L)\cap G\ll G$. As $(H+G)\cap L\leq[(G+L)\cap H]+[(H+L)\cap G]$, $H+G$ is a supplement of L in A. □

Theorem 2.10. For any semiring S, any finite direct sum of $\bigoplus$-supplementd S-semimodules is $\bigoplus$-supplementd.

Proof: Lease m be a positive integer besides A_i be a $\bigoplus$-supplemente S-semimodule for all $1\leq i\leq m$. Let $A=A_1\bigoplus\cdots\bigoplus A_m$. To show that A is $\bigoplus$-supplemente it is sufficient by induction on m to show this is the case when $m=2$. So, take $m=2$.
Let $L\leq A$. Then $A=A_1+A_2+L$ thus that A_1+A_2+L has a supplement 0 in A. Let H be a supplement of $A_2\cap(A_1+L)$ in A_2 with H is a direct summand of A_2. Using Lem 2.9, H is a supplement of A_1+L in A. Lease K be a supplement of $A_1\cap(L+H)$ in A_1 with K is a direct summand of A_1. For a second time applying Lem 2.9, we get that $H+K$ is a suplement to L in A. Since $H\leq_{\bigoplus}A_2$ and $K\leq_{\bigoplus}A_1$ so, $H+K=H\bigoplus K\leq_{\bigoplus}A$. As a result $A=A_1\bigoplus A_2$ is $\bigoplus$-supplemented. □

Corollary 2.11. A finite direct sum of semimodules with (D_1) is $\bigoplus$-supplementd.

Corollary 2.12. Any finite direct sum of hollow (or local) semimodules is $\bigoplus$-supplementd.

Example 2.13.
(1) Consider $\mathbb{N}_0$ is the semiring of non-negative integers. As $\mathbb{N}_0$ is a local $\mathbb{N}_0$-semimodule. Now by Corollary 2.12, $\mathbb{N}_0$ is $\bigoplus$-supplemented $\mathbb{N}_0$-semimodule.
(2) Consider $\mathbb{Z}_{p^n}$ as an $\mathbb{Z}$-semimodule where p is prime number and $n\in\mathbb{N}$. Now by Corollary 2.12, $\mathbb{Z}_{p^n}$ is $\bigoplus$-supplemented.

A commutativ semiring S is named a valuation semiring if it is a local semiring besides each finitely generated ideal is principal [6]. A semimodule A is named finitely presented if $A=\frac{F}{N}$ for certain finitely generated free semimodule F besides finitely generated subsemimodule N of F.
Similar to [9, Example 2.2] we have the next example show this a factor semimodule of a $\bigoplus$-supplemented semimodule is not in general $\bigoplus$-supplemented.

Example 2.14. Assume S is a commutativ local semiring which is not a valuation semiring. As in [9, Example 2.2], there is an indecomposable finitely presented semimodule $A=\frac{S^{(n)}}{K}$, which cannot be generated by fewer than n elements. Using [9] $S^{(n)}$ is $\bigoplus$-supplemente, $n\in\mathbb{N}$. However A is not $\bigoplus$-supplemente.
Theorem 2.16 deals with a special case of factor semimodules of $\bigoplus$-supplemented semimodules. First, we show the next lemma.

Lemma 2.15. Assume A is a semimodule besides let $U\leq A$ such that $f(U)\leq U$ for every $f\in End_S(A)$. If $A=A_1\bigoplus A_2$, then $U=U\cap A_1\bigoplus U\cap A_2$.

Proof: Assume $\pi_i:A\to A_i$ $(i=1,2)$ indicate the canonical projections. Take $x\in U$. Now $x=\pi_1(x)+\pi_2(x)$. Using supposition, $\pi_i(U)\leq U$ for $i=1,2$. Hence $\pi_i(x)\in U\cap A_i$ for $i=1,2$. Thus $U\leq U\cap A_1\bigoplus U\cap A_2$. Hence $U=U\cap A_1\bigoplus U\cap A_2$. □

Theorem 2.16. Let A be a subtractive semimodule besides let $U\leq A$ with $f(U)\leq U$ for all $f\in End_S(A)$. If A is $\bigoplus$-supplemented, at that time A/U is $\bigoplus$-supplemented. If, also, U is a direct summand of A, at that time U is also $\bigoplus$-supplemented.

Proof: As A is a subtractive S-semimodule, we get A/U is an S-semimodule [7, p. 165]. Assume A is a $\bigoplus$-supplemented semimodule. Let L be a sub-semimodule of A which contains U. There is $N,N'\leq A$ with $A=N\bigoplus N'$, $A=L+N$, and $L\cap N\ll N$. By [16], Lem. 1.2(d)], $(N+U)/U$ is a supplement of L/U in A/U. Currently apply Lem. 2.15 to have this $U=U\cap N\bigoplus U\cap N'$. As a result,

$$(N+U)\cap(N'+U)\leq(N+U+N')\cap U+(N+U+U)\cap N'$$

So,

$$(N+U)\cap(N'+U)\leq U+(N+U\cap N+U\cap N')\cap N'$$

From now $(N+U)\cap(N'+U)\leq U$ and $((N+U)/U)\bigoplus((N'+U)/U)=A/U$. Now $(N+U)/U$ is a direct summand of A/U. Therefore, A/U is $\bigoplus$-supplemented.
At the present asume U is a direct summand to A. Let V be a subsemimodule in U. As A is $\bigoplus$-supplemented, there exist K, $K'\leq A$ with $A=K\bigoplus K'$,

$A = V + K$, and $V \cap K \ll K$. Hence $U = V + U \cap K$. However $U = U \cap K \bigoplus U \cap K'$ by Lem. 2.15, hereafter $U \cap K$ is a direct summand of U. As well, $V \cap (U \cap K) = V \cap K \ll K$. Now, $V \cap (U \cap K) \ll U \cap K$ by [16, Lem. 1.1(b)]. So $U \cap K$ is a supplement of V in U besides it is a direct summand of U. Henceforth U is $\bigoplus$-supplementd. □

Corollary 2.17. Assume A is a subtractive S-semimodule besides $P(A)$ the sum of all its radical subsemimodules. If A is $\bigoplus$-supplemente, at that time $A/P(A)$ is $\bigoplus$-supplemente. If, furthermore, $P(A)$ be a direct sumand to A, at that time $P(A)$ is also $\bigoplus$-supplemented.

3. Completely $\bigoplus$-supplemented semimodules

Even though the properties lifting (or D_1), amply supplementd besides supplementd are inherited by summands, it is unknown (and improbable) that the same is correct for the property $\bigoplus$-supplemented since it is not true in modules as in [8].

Similar to [8] we give the next definition of completely $\bigoplus$-supplemente semimodules.

Definition 3.1. A semimodule A is named completely $\bigoplus$-supplemented if every direct summand of A is $\bigoplus$-supplemented.
Remarked that an S-semimodule A is supplemented if and only if A is $\bigoplus$-supplemented whenever S is Dedekind semidomain. Thus an S-semimodule A is $\bigoplus$-supplementd if and only if A is completely $\bigoplus$-supplementd. For more information about semidomains, see [4,6].
Clearly, every lifting (or D_1) semimodule is completely $\bigoplus$-supplemented.

Example 3.2. Assume x is any integer besides indicate A the $\mathbb{Z}$-semimodule $(\mathbb{Z}/x^i\mathbb{Z}) \bigoplus (\mathbb{Z}/x^j\mathbb{Z})$ $(i, j \in \mathbb{N})$. At that time A is completely $\bigoplus$-supplemented (see [8, Example 2.16]).

Definition 3.3. Given a positive integer m, the semimodules A_i $(1 \le i \le m)$ are named relatively projective if A_i is A_j- projective for all $1 \le i \ne j \le m$.

Proposition 3.4. [7, Proposition 14.22] (Semimodularity Law) Let A be a semimodule over semiring S besides let N and K be subsemimodules of A. Let L be a subtractive subsemimodule of A with $N \subseteq L$. At that point $L \cap (N + K) = N + (L \cap K)$.

Theorem 3.5. Let A_i $(1 \le i \le m)$ be a finite collection of relatively projective subtractive semimodules. Now the semimodule $A = A_1 \bigoplus \cdots \bigoplus A_m$ is $\bigoplus$-supplementd iff A_i is $\bigoplus$-supplementd for each $1 \le i \le m$.

Proof: The sufficiency is showed in Thm 2.10. In opposition, we just show A_1 to be $\bigoplus$-supplemented. Let $F \le A_1$. Now there is $K \le A$ with $A = F + K$, K is a direct sumand to A besides $F \cap K \ll K$. Since $A = F + K = A_1 + K$, by [10, Lemma 4.47], there exists $K_1 \le K$ such that $A = A_1 \bigoplus K_1$. Now $K = K_1 \bigoplus (A_1 \cap K)$ by using Proposition 3.4, since $K_1 \le K$ and K is a subtractive subsemimodule of A. Note that $A_1 = F + (A_1 \cap K)$ and $A_1 \cap K$ is a direct sumand to A_1. Henceforth, $F \cap K = F \cap (A_1 \cap K) \ll A_1 \cap K$ as in modules see [10, Lemma 4.2]. Thus A_1 is $\bigoplus$-supplement. □

Theorem 3.6. Let A be a $\bigoplus$-supplemented semimodule with (D_3). At that time A is completely $\bigoplus$-supplemented.

Proof: Assume N is a direct sumand to A besides $F \le N$. We show F has a supplement in N that is direct sumand of N. As A is $\bigoplus$-supplemente, there exists a direct summand K of A with $A = F + K$ besides $F \cap K \ll K$. As a result $N = F + (N \cap K)$. Moreover, $N \cap K$ is a direct sumand of A has (D_3). Now $F \cap (N \cap K) = F \cap K \ll N \cap K$. □

Definition 3.7. [1] Let A be a semimodule. A subsemimodule N of A is closed in A if N has no proper essential extensions in A.

Definition 3.8. [1] A semimodule A is named extending semimodule if every closed subsemimodule of A is a direct summand of A. A is said to be extending (CS-semimodul) if every subsemimodul of A is essential in a direct summand of A.
In [11] P. F. Smith calls a module A is named UC-module if each submodule of A has a unique closure in A. Similar to [11], we have the next definition.

Definition 3.9. A semimodule A is named UC-semimodule if each subsemimodul of A has a unique closure in A.

Lemma 3.10. Let A be a UC extending semimodule. Then A has (D_3).

Proof: Assume A_1, A_1 are direct summands of A with $A = A_1 + A_2$. Using [15], Proposition 1.1], $A_1 \cap A_2$ is a closed subsemimodule of A. As A is extending, $A_1 \cap A_2$ is a direct summand of A. As a result A has (D_3). □

Proposition 3.11. Assume A is a UC extending semimodule. Now A is $\bigoplus$-supplemented iff A is completely $\bigoplus$-supplemented.

Proof: The sufficiency is evidence. Conversly, supposing A is $\bigoplus$-supplemente. Using Lemma 3.10, A has (D_3). Thus A is completely $\bigoplus$-supplemente from Theorem 3.6. □

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