Cancellation and Weak Cancellation S-Acts

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Cancellation and Weak Cancellation S-Acts

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Abstract

The purpose of this work is to introduce and study a generalization of the cancellation (weak cancellation) of modules to acts. Then some notions related of modules was extend to arbitrary acts. Several statements that are equivalent to the cancellation (weak cancellation) property have been studied. Some characterizations of cancellation (weak cancellation) acts was presented and derive many of their properties. Using the condition of multiplication acts various results of cancellation (weak cancellation) acts are investigated.

Keywords: Cancellation acts, Cyclic acts, Finitely generated act, Multiplication acts, Weak cancellation acts

1. Introduction

Let S be a semigroup. Then a nonempty set A is called left S-act if there exists a mapping: \(S \times A \rightarrow A\) written by \((s,a) \rightarrow sa\), satisfying \((s_1s_2)a = s_1(s_2a)\) for all \(s_1, s_2 \in S\) and \(a \in A\) \([1]\). In a similar manner, the definition of right S-acts. The term “S-act” will henceforth always refer to the left S-act. A semigroup with identity is called a monoid. A nonempty subset \(B\) of S-act A is said to be subact (briefly, S-subact) if for all \(s \in S\) and \(b \in B\), then \(sb \in B\). An element \(\theta \in A\) is said to be fixed of A if \(s\theta = \theta\) for all \(s \in S\). Through this paper an S-act A has an unique fixed element such that if a semigroup S containing zero element such that \(0a = \theta = s0\) for every \(a \in A, s \in S\) then \(0\) will be named the zero of A. Let A be a S-act and \(U\) be a nonempty subset of A. Then \(A\) is a S-act if \(U\) is a S-subact of A and \(U\) be a nonempty subset of A. If for all element \(a \in A\) can be represented as \(a = su\) for some \(u \in U, s \in S\) then U a generating set of A. In other words, U is a generating elements of A if \(\langle U \rangle = \bigcup_{u \in U} Su = A\). An S-act A is cyclic if \(A = \langle a \rangle\) where \(a \in A\). The set U is said to be a basis of S-act A if it is generating elements of A and every element \(a \in A\) has a unique representation in the form \(a = us\) where \(u \in U\) and \(s \in S\), i.e., if \(a = u_1s_1 = u_2s_2\) then \(u_1 = u_2\) and \(s_1 = s_2\).

Let S be a semigroup and \(\rho\) be a equivalence relation on S. Then \(\rho\) is a congruence relation on A if \(a'\rho b'\) implies \((sa)\rho(s'a')\) for all \(a, a' \in A\). An equivalence relation \(\rho\) on a semigroup is called a congruence if and only if \(s\rho t\) and \(u\rho v\) implies \((su)\rho(tv)\) for all \(s, t, u, v \in S\). In particular, let A be a left S-act and \(\rho\) be a equivalence relation on A then \(\rho\) is a congruence relation on A if \(a\rho a'\) implies \((sa)\rho(s'a')\) for all \(a, a' \in A\) and \(s \in S\). Let B be an S-subact of S-act A and \(\rho\) is congruence on A, then \(\rho_B\) is a congruence on A such that \(\rho_B = \rho \cap B \times B\). For equivalence relation \(\rho\) the class of \(s \in S\) with respect to \(\rho\) is indicated by \([s]_\rho\). For a congruence \(\rho\) on S the multiplication in \(S/\rho\) is written by \([s]_\rho [t]_\rho = [st]_\rho\) for all \(s, t \in S\). Thus \(S/\rho\) is a semigroup that called the factor semigroup of S by \(\rho\). Let A be a S-act and \(\rho\) be a congruence on A. Then \(A/\rho = \{[a]_\rho \mid a \in A\}\) and a left multiplication by elements of S on \(A/\rho\) is defined by \(s[a]_\rho = [sa]_\rho\) for every \(s \in S\). Let S and \(S'\) be semigroups. The mapping \(f : S \rightarrow S'\) is said to be a semigroup homomorphism of S into \(S'\) if \(f(s_1s_2) = f(s_1)f(s_2)\) for each \(s_1, s_2 \in S\). In this case \(ker f = \{[s_1, s_2] \in S \times S \mid f(s_1) = f(s_2)\}\) is said to be kernel congruence of homomorphism \(f\). Let S be a semigroup and \(\rho\) be a congruence on S. The mapping \(\pi : S \rightarrow S/\rho\) is defined by \(s \mapsto [s]_\rho\) is a homomorphism and it is called the canonical epimorphism \([1]\). For S-act A the ideal \(\{s \in S \mid sa = \theta\} \text{ for all } a \in A\) of S is said to be the annihilator of S-act A in S (briefly, ann(A)). If ann(A) = 0, A is called faithful. Let B be a S-subact of S-act A. Then \(B : A = \{s \in S \mid sa \in B\} \text{ for each } a \in A\). Clearly, \(B : A\) is ideal of a semigroup S [2,3]. An S-act A is called a multiplication if for all S-subact B of A, there exists an ideal \(K\) of a semigroup S such that \(B = KA [4,5]\).

The main aim of this paper is to study the concept of cancellation (weak cancellation) S-acts as a generalization of cancellation (weak cancellation) modules that was introduced by Adil GN, Ali SM...
Thus some results and concepts of modules was extend to arbitrary S-acts, analogous to the notions of cancellation (weak cancellation) modules introduced by Adil GN, Ali SM [6,8]. Several results and characterizations about these concepts have been studied. In particular, some results and equivalent conditions for this concept was given such as: an S-act is cancellation if and only if it's weak cancellation and faithful S-act. Also has been proven that the homomorphic image of cancellation S-act need not be a cancellation but the inverse image is hold. On other hand, it has been proven that if M is cancellation S-act, then it is finitely generated and several related notions was investigated to establish a connection between multiplication S-acts and cancellation S-acts. The relation between cancellation S-acts and some types of S-acts was discussed. Finally, in this paper S means a semigroup with zero element and the word “ideal” is used for a two sided ideal.

2. Main results

The concept of cancellation (weak cancellation) was introduced by Adil GN, Ali SM [6,8] as follows: if M is an R-module, where R is a commutative ring with unity. Then M is said to be a cancellation R-module (resp. weak cancellation R-module) if CM = DM for ideals C, D of R, then C = D (resp. C + ann(M) = D + ann(M)). A natural generalization of this concept to S-acts was studied.

Definition 2.1:
An S-act A is said to be cancellation S-act if CA = DA for ideals C, D of a semigroup S then C = D. An ideal K of a semigroup S is a cancellation if it is cancellation S-subact of S-act S.

Examples 2.2.

i. Let $S = A = \{a, b, c\}$. Then A is an S-act under the mapping $S \times A \rightarrow A$ that is defined by the following table:

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Clearly, A is cancellation S-act.

ii. Let $S = A = \{a, b, c\}$. Define the mapping $S \times A \rightarrow A$ by the following table:

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Then A is a S-act. Now, $K = \{a\}$, $L = \{a, b\}$, $C = \{a, c\}$ and $D = S$ are ideals of $S$. Here, $AM = CM = \{a\}$ but $A \neq C$. Thus M is not cancellation S-act.

iii. If $\mathbb{Z}$, $\mathbb{Q}$ are the sets of all integer and rational numbers, respectively. Then $\mathbb{Q}$ is an $\mathbb{Z}$-act. It's clear that $(p\mathbb{Z})\mathbb{Q} \subseteq \mathbb{Q}$. If $q \in \mathbb{Q}$, then $q = \frac{1}{k} = \frac{p}{kq} = p\frac{x}{y} \in (p\mathbb{Z})\mathbb{Q}$ such that $x, y \in \mathbb{Z}$. Thus $\mathbb{Q}$ is not cancellation $\mathbb{Z}$-act.

Remark 2.3.

An S-act A is a cancellation if and only if it is a faithful.

Proposition 2.4.
Let K be an ideal of a semigroup S and A be a cancellation S-act. Then KA is a cancellation S-subact if and only if K is cancellation ideal.

Proof: $(\Rightarrow)$ Let $DK = CK$, where D and C are ideals of S. Then $(DK)A = (CK)A$ and since KA is a cancellation then $D = C$.

$(\Leftarrow)$ Let $D(KA) = (CA)$, where D and C are ideals of S. Since A is cancellation then $DK = CK$ and by hypothesis $D = C$.

A be S-act. An element $a \in A$ is called a torsion if there exist $r, s \in S$ where $r \neq s$ such that $ra = sa$. Otherwise, $a$ is non-torsion element.

Proposition 2.5.
Every cyclic S-act generate by non-torsion element is a cancellation S-act.

Proof: Suppose that $a \in A$ is non-torsion element and C, D are ideals of S, where $C < A > = D < A >$. Then $ca \in D < a >$ for all $c \in C$ thus there exists $d \in D$ such that $ca = da$. Since $a$ is non-torsion $c = d$. Therefore, $C \subseteq D$. In the same way, $D \subseteq C$. Thus $C = D$.

The following example, shows that the condition of generated M by non-torsion element is necessary: Let $\mathbb{Z}_2$ be $\mathbb{Z}$-act. Then it's clear that $\mathbb{Z}_2 = (1)$. Now, $(2)$ $\mathbb{Z}_2 = 0$ and $(0)\mathbb{Z}_2 = 0$. Thus $(2)\mathbb{Z}_2 = (0)\mathbb{Z}_2$ but $(2) \neq (0)$.

The following Proposition gives an equivalent of cancellation property:

Theorem 2.6.
Let $S$ be a semigroup. Then the following are equivalent for an S-act $A$.

i. $A$ is cancellation S-act.

ii. If $C$ and $D$ are ideals of $S$ such that $CA \subseteq DA$, then $C \subseteq D$.

iii. If $s \in S$ and $D$ is ideal of $S$ such that $(s)A \subseteq DA$, then $s \in D$.

iv. Let C be an ideal of S. Then $[CA:A] = C$.

Proof: (i) $\Rightarrow$ (ii) Let $CA \subseteq DA$. Then $DA = CA \cup DA = (C \cup D)A$ and by hypothesis, $D = C \cup D$. Thus $C \subseteq D$. 
(ii) ⇒ (iii) Suppose \( s \in S \) and \( D \) is ideal of \( S \) where \( sA \subseteq CA \). Then \( sC \subseteq D \), and hence \( s \in D \).

(iii) ⇒ (iv) Let \( x \in [CA: A] \). Then \( x A \subseteq CA \) thus \( x \in C \). Conversely, let \( y \in C \) then \( y A \subseteq CA \). So \( y \in [CA: A] \). Hence, \([CA:A] = C\).

(iv) ⇒ (i) Suppose \( CA = DA \) where \( C, D \) are ideals of \( S \). Then \( CA \subseteq DA \) and hence \( C \subseteq [DA:A] = D \). In the same way \( D \subseteq C \). Thus, \( C = D \).

Proposition 2.7.
Let \( A \) be a \( S \)-act over a semigroup \( S \). Then for all ideals \( C \) and \( D \) of \( S \), \( [C:D] = [CA:DA] \) if and only if \( A \) is a cancellation.

Proof: (⇒) Let \( s \in [C:D] \) then \( s \in D \subseteq C \). Thus, \( s \in [CA:DA] \).

Proof: (⇐) Assume that \( DA \subseteq CA \) whenever \( C, D \) are ideals of a semigroup \( S \). Then \([CA: DA] = S \) and hence \([C:D] = S \), thus \( D \subseteq C \). By Theorem 2.6, \( A \) is a cancellation \( S \)-act.

Proposition 2.8.
Every free \( S \)-act is a cancellation.

Proof: Suppose \( A \) is a free \( S \)-act with basis \( \{x_1, x_2, x_3, \ldots \} \) and \( C, D \) be ideals of \( S \) where \( CA \subseteq DA \). Let \( c \in C \). Then \( cx_1 \in DA \). Hence \( cx_1 = u_{i=1}^n d_i x_i \).

Thus, \( d_i = 0 \) for all \( i \neq 1 \). Says that \( c \in D \). So \( C \subseteq D \). By Theorem 2.6, \( M \) is a cancellation \( S \)-act.

Suppose \( C \subseteq D \) then \( C \subseteq [DA:A] \). Conversely, let \( DA \subseteq CA \) whenever \( C, D \) are ideals of \( S \). Then \([CA: DA] = S \) and hence \([C:D] = S \), thus \( D \subseteq C \).

Theorem 2.9.
Let \( \{B_i\}_{i \in I} \) be a collection of ideals of a semigroup \( S \) and \( A \) be a \( S \)-act. Then \( \bigcup_{i \in I} (B_i A) = (\bigcup_{i \in I} B_i) A \).

Proof: Clear.

If \( \{f_i: A \rightarrow B; A, B \text{ are } S \text{-acts } \} \) is a collection of homomorphisms, then \( \mathcal{F} = \bigcup_{i \in I} f_i (A) \), is a notation it's used throughout this work, where \( \mathcal{F} \) is an \( S \)-subact of \( B \). If \( B=S \), then the symbol \( \mathcal{F} \) is used instead of \( \mathcal{F} \).

The following Theorem proves that the inverse image of a cancellation \( S \)-acts is a cancellation.

Theorem 2.10.
Let \( A, B \) be \( S \)-acts and \( \mathcal{F} \) be a cancellation \( S \)-subact of \( B \). Then \( A \) is a cancellation.

Proof: If \( C, D \) are ideals of a semigroup \( S \) such that \( CA = DA \), then \( f_i (CA) = f_i (DA) \) and hence \( \bigcup_{i \in I} f_i (CA) = \bigcup_{i \in I} f_i (DA) \).

But \( f_i (CA) = f_i (A) \) and \( f_i (DA) = D f_i (A) \). Thus, \( C \bigcup_{i \in I} f_i (A) = \bigcup_{i \in I} f_i (CA) = \bigcup_{i \in I} f_i (DA) = D \bigcup_{i \in I} f_i (A) \).

So \( C \mathcal{F} = D \mathcal{F} \). Hence \( C = D \).

Corollary 2.11.
If \( A \) is a \( S \)-act and \( \mathcal{F} \) is a cancellation ideal of a semigroup \( S \). Then \( A \) is a cancellation.

Corollary 2.12.
Let \( B, A \) be \( S \)-acts where \( B \) is a homomorphic image of \( A \). If \( B \) is a cancellation. Then \( A \) is a cancellation.

Proposition 2.13.
If \( \rho \) is a congruence on \( S \)-act \( A \), then \( A \) is cancellation if and only if \( A/\rho \) is a cancellation \( S \)-act.

Proof: \((⇒)\) Let \( C \) and \( D \) be ideals of a semigroup \( S \), such that \( C \{A/\rho\} = D \{A/\rho\} \). Let \( c \in C \). Then \( c [a]_\rho \in C \{A/\rho\} \).

Thus, there is \( d \in D \) and \( [a']_\rho \in [A/\rho] \) such that \( c [a]_\rho = d [a']_\rho \). Hence \( [ca]_\rho = [da']_\rho \). Thus \( ca = da' \in DA \).

By Theorem 2.6, \( c \in D \). Then \( C \subseteq D \). Similarly, \( D \subseteq C \). Therefore, \( A/\rho \) is a cancellation.

\((⇐)\) Since \( A/\rho \) cancellation \( S \)-act and it is homomorphic image of \( A \) then by Corollary 2, \( A \) is a cancellation.

Now, another type of cancellation properties was introduced, it's called weak cancellation property that it's satisfy all cyclic act.

Definition 2.14.
Let \( A \) be an \( S \)-act. Then \( A \) is said to be a weak cancellation whenever if \( C, D \) are ideals of \( S \) such that \( CA = DA \), then \( C \cup \text{ann}(A) = D \cup \text{ann}(A) \).

Example 2.15.
Let \( S = A = \{a, b, c, d\} \). Define a mapping: \( S \times A \rightarrow A \) by \( (C, D) \mapsto C \cap D \). Then \( A \) is a \( S \)-act. Hence, \( A \) is weak cancellation.

Remark 2.16.

i. It's clear that cancellation \( S \)-act is weak cancellation. But the following example proves that the converse is not hold:

Let \( S = A = \{a, b, c, d\} \). Then \( A \) is a \( S \)-act by the mapping: \( S \times A \rightarrow A \) which defined by the following table:

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Here, \( k_1 = \{a\} \), \( K_2 = \{a, b\} \), \( K_3 = \{a, c\} \), \( K_4 = \{a, b, c\} \), \( K_5 = \{a, c, d\} \) and \( K_6 = S \) are the ideals of \( S \). Here, \( A \) is weak cancellation \( S \)-act. On other hand, \( K_2 M = K_4 M \) but \( K_2 \neq K_4 \). Thus, \( M \) is not cancellation \( S \)-act.

ii. An \( S \)-act. \( A \) is a cancellation if and only if \( i \) it is weak cancellation and faithful.
By specific condition the following Proposition shows that every weak cancellation S-act is cancellation.

**Proposition 2.17.**
Let A be a weak cancellation S-act and \( \text{ann}(A) \subseteq K \) for all ideal K of a semigroup S. Then M is cancellation S-act.

**Proof:** Clear.

**Proposition 2.18.**
Every cyclic S-act generated non-torsion element is weak cancellation.

**Proof:** It's analogous to the proof of Proposition 2.5.

Now, in the following Theorem a characterization of weak cancellation S-acts was given.

**Theorem 2.19.**
Let A be a S-act and S be a semigroup. Then the following are equivalent.

i. A is a weak cancellation S-act.
ii. If C, D are ideals of S with \( C \subseteq D \), then \( C \subseteq D \cup \text{ann}(A) \).
iii. If \( d \in S \) and D is ideal of S such that \( (d)A \subseteq DA \), then \( d \in D \cup \text{ann}(A) \).
iv. If C is ideal of S and \( \text{ann}(A) \subseteq C \) then \( [C:A] = C \).

**Proof:** (i) \( \Rightarrow \) (ii) Let C, D be ideals of a semigroup S such that \( C \subseteq DA \). Then \( DA = CA \cup DA = (C \cup D)A \). By hypothesis \( D \cup \text{ann}(A) \). Hence \( C \subseteq D \cup \text{ann}(A) \).

(ii) \( \Rightarrow \) (iii) Let \( (d)A \subseteq DA \). Then \( d \in D \cup \text{ann}(A) \) and hence \( d \in D \cup \text{ann}(A) \).

(iii) \( \Rightarrow \) (iv) Let \( x \in [C:A] \). Then \( xA \subseteq C \). By hypothesis, \( x \in \text{C} \cup \text{ann}(A) \). Thus \( x \in C \). Conversely, let \( y \in C \) then \( yA \subseteq C \). Hence \( y \in [C:A] \).

(iv) \( \Rightarrow \) (i) Suppose \( CA = DA \) where \( C, D \) are ideals of a semigroup S. Then \( CA \subseteq DA \). Hence \( C \subseteq [DA:A] = D \). Hence, \( C \cup \text{ann}(A) \subseteq D \cup \text{ann}(A) \). Similarly, \( D \cup \text{ann}(A) \subseteq C \cup \text{ann}(A) \). Thus, \( C \cup \text{ann}(A) = D \cup \text{ann}(A) \).

**Proposition 2.20.**
Let S be a semigroup. An S-act A is a weak cancellation if and only if \([C:D] = [CA:DA] \) for all ideals C and D of semigroup S, such that \( \text{ann}(A) \subseteq C \).

**Proof:** \( \Rightarrow \) Let \( x \in [C:D] \) then \( xD \subseteq C \) and hence \( xDA \subseteq CA \). So \( x \in [CA:DA] \). Conversely, let \( y \in [CA:DA] \) then \( yD \subseteq CA \). By Theorem 2.19, \( yD \cup \text{ann}(A) = C \). Hence \( y \in [C:D] \).

\( \Leftarrow \) Suppose that \( DA \subseteq CA \). Then \( [CA:DA] = S \) and hence \([C:D] = S \) implies \( D \subseteq C \cup \text{ann}(A) \). Thus, \( D \subseteq C \cup \text{ann}(A) \).

By Theorem 2.19, A is a weak cancellation S-act.

The example below shows that the homomorphic image of a weak cancellation S-act not necessary a weak cancellation S-act.

**Example 2.21.**
If \( A = S = \mathbb{Z} \) and \( B = \{1, 2\} \), then A is a S-act by the usual multiplication of integer numbers and B is an S-act by a mapping: \( S \times B \rightarrow B \) written by \((s, b) \rightarrow 2\) for all \( s \in S, b \in B \). It's clear that A is \( a^* \) weak cancellation. For S-act B, if \( C = uS \) and \( D = vS \) are ideals of S where \( u \neq v \in S \) such that \( CB = DB = \{2\} \).

But \( C \neq D \) thus B is not cancellation S-act. On other hand, define a mapping \( f: A \rightarrow B \) by:

\[
f(a) = \begin{cases} 
1 & \text{if } a \text{ is odd} \\
2 & \text{if } a \text{ is even}
\end{cases}
\]

Here, \( f \) is a homomorphic image of A. Therefore, A is a weak* cancellation. But B is not a weak cancellation S-act (note that B is faithful).

A similar result to Theorem 2.10, will be shown in the next Theorem for the property of weak cancellation.

**Theorem 2.22.**
Let A and B be S-acts. If \( \mathcal{F} \) is a weak cancellation S-subact of B and \( \text{ann}(\mathcal{F}) = \text{ann}(A) \), then A is weak cancellation S-act.

**Proof:** Let \( CA = DA \) such that \( C, D \) are ideals of S. Then \( \bigcup_{i \in I} f_i(CA) = \bigcup_{i \in I} f_i(DA) \). But \( C \cup \bigcup_{i \in I} f_i(A) = D \) and \( \bigcup_{i \in I} f_i(A) \). So, \( C, \mathcal{F} \subseteq D \mathcal{F} \). Since \( \mathcal{F} \) is cancellation.

Then \( Cuann(\mathcal{F}) = Duann(\mathcal{F}) \) and hence \( Cuann(A) = Duann(A) \).

**Corollary 2.23.**
Let A be a S-act and \( \mathcal{F}_S \) be a weak cancellation ideal of a semigroup S, where \( \text{ann}(\mathcal{F}_S) = \text{ann}(A) \). Then A is weak cancellation S-act.

**Corollary 2.24.**
Let A, B be a S-acts, where B is a homomorphic image of A. If B is a weak cancellation S-act and \( \text{ann}(B) = \text{ann}(A) \), then A is a weak cancellation S-act.

**Proposition 2.25.**
Let \( \rho \) be a congruence on S-act A. Then A is weak cancellation if and only if \( A/\rho \) is weak cancellation S-act.

**Proof:** Clear.

**Theorem 2.26.**
Let S be a semigroup and A be a S-act with \( S = S/\rho \) where \( \rho \) is a congruence on S. If A is a cancellation S-act, then A is a weak cancellation S-act. The converse is true if A is faithful.

**Proof:** Suppose that \( KA = DA \) such that K and D are ideals of S. Let \( k \in K \) and \( a', a \in A \) then \( ka = da' \) where \( d \in D \). Then \( [k]_\rho [a] = [d]_\rho [a'] \). Let \( \pi : S \rightarrow S/\rho \) be a canonical mapping such that \( \pi(D) = D \Rightarrow \pi(D)A = D \). Thus \( [d]_\rho A = [a']_\rho A \) and hence \( [k]_\rho A = [a]_\rho A \). By Theorem 2.6, \( [k]_\rho \in D \) then there is \( d' \in D \) such that \( [k]_\rho = [d]_\rho = [d']_\rho \).
Thus $k = d' \in D$. So KD. Similarly, $D \subseteq K$ and hence $D = K$. Therefore, $K \cup \text{ann}(A) = D \cup \text{ann}(A)$.

Conversely, suppose $A$ is a weak cancellation $S$-act and $B$ and $C$ are ideals of a semigroup $S$, where $BA = CA$. Then $B = \omega / \rho w$ and $C = L / \rho l$, such that $\omega$ and $L$ are ideals of $S$. Let $w \in W$ then $[w]_{\rho w} \in B$. Thus there exists $l \in L$ and $m' \in M$ where $[w]_{\rho w} = [l]_{\rho l} \Rightarrow [w]_{\rho w} = [m']_{\rho w} \Rightarrow wm = lm$. This implies that $wm \in L$. By Theorem 2.19, $w \in L \cup \text{ann}(A)$. By faithfulness of $A$, $w \in L$. Then $W \subseteq L$. Similarly, $L \subseteq W$. Hence $W = L$. Therefore, $W / \rho w = L / \rho l$ and hence $B = C$. Thus $A$ is cancellation $S$-act.

Now, our attention turn to the concept of cancellation $S$-acts in a multiplication $S$-acts. Before this the following notion was needed in our work:

For $S$-act $A$, define $\tau(A) = \bigcup_{a \in A} [Sa : A]$. It’s clear that $\tau(A)$ is ideal of a semigroup $S$. In case of $A$ is ideal of a semigroup $S$, then $A \subseteq \tau(A)$.

The following Proposition was introduced to be useful in studying of cancellation (weak cancellation) $S$-acts.

**Proposition 2.27.**

If $S$ is a commutative semigroup and $A$ is a multiplication $S$-act, then $A = \tau(A)$ and $A \subseteq \tau(A)$. More generally for all $S$-subact $B$ of $A$, $B = \tau(B)$.

**Proof:** Suppose $a \in A$, then $S = [Sa : A]$. Thus, $A = \bigcup_{a \in A} [Sa : A] = \bigcup_{a \in A} \bigcup_{[Sa : A]} A = \tau(A)$.

Let $B$ be a $S$-subact of $A$. Then there exists an ideal $K$ of a semigroup $S$ such that $B = KA$. Hence, $B = KA = K(\tau(A)) = \tau(A)[K]A = \tau(A)B$.

**Proposition 2.28.**

Let $S$ be a monoid and $A$ be a multiplication $S$-act. If $A$ is a cancellation, then $A$ is finitely generated.

**Proof:** By hypothesis, $A = \tau(A)$ and $\tau(A) \subseteq A$. Thus, $A = \tau(A)$ and hence $A = \tau(A) \subseteq A$. Thus, $A = \tau(A)$ and hence $A = \tau(A)$.

\[\begin{align*}
\text{Therefore, } & \tau(A) = A. \\
\text{Proposition 2.29.} & \\
\text{Let } S \text{ be a monoid and } A \text{ be a multiplication } S\text{-act. If } A \text{ is a weak cancellation, then } \tau(A) \text{ is finitely generated.} \\
\text{Proof:} & \text{ Assume that } A \text{ is a multiplication } S\text{-act, then } A = \tau(A) \subseteq A. \text{ Since } \text{ann}(A) \subseteq \tau(A), \text{ implies that } \tau(A) = S. \text{ Thus, } S \text{ is finitely generated.} \\
\text{Proposition 2.30.} & \text{ Let } S \text{ be a commutative monoid and } B \text{ be a weak cancellation } S\text{-subact of multiplication } S\text{-act } A \text{ with } \text{ann}(A) = \text{ann}(B). \text{ Then } A \text{ is weak cancellation.} \\
\text{Proof:} & \text{ Assume that } A \text{ is a multiplication } S\text{-act. Then } B = KA \text{ for some ideal } K \text{ of a semigroup } S. \text{ Now, let } CA = DA \text{ for ideals } C, D \text{ of a semigroup } S.
\]

Then, $C(KA) = D(KA)$. Thus $CB = DB$. So, $C \cup \text{ann}(B) = D \cup \text{ann}(A)$. Therefore, $C \cup \text{ann}(A) = D \cup \text{ann}(A)$. Moreover, $A$ is finitely generated (Proposition 2.29).

**Corollary 2.31.**

If $S$ is a commutative monoid, and $B$ is a cancellation $S$-subact of a multiplication $S$-act $A$, then $A$ is cancellation.

**Proof:** Since $B$ is a cancellation $S$-subact of $A$, $B$ is a weak cancellation and $\text{ann}(B) = 0$ (By Remark 2.16 (ii)). Then, $\text{ann}(B) = \text{ann}(A) = 0$ and by Proposition 2.30, $A$ is a weak cancellation and $\text{ann}(A) = 0$. Hence, $A$ is cancellation.

**Proposition 2.32.**

Let $S$ be a monoid and $A$ be a multiplication $S$-act. If $A$ is weak cancellation, then it is finitely generated.

**Proof:** Clear.

**Proposition 2.33.**

Let $K$ be ideal of a semigroup $S$ and $B$ be a $S$-subact of a multiplication $S$-act $A$. If $A$ is a cancellation, then $[KB : A] = [KB : A]$.

**Proof:** $s \in K[B : A]$, then $s = kr$ such that $k \in K$ and $r \in [B : A]$. Thus $r A \subseteq B \Rightarrow K(r A) \subseteq KB \Rightarrow s A = (kr) A \subseteq KB$.


3. Conclusions

The main idea is to introduce the concept of a cancellation acts. Thus some results, properties and Theorems were discussed. Many of characterizations about these concept and its properties was given. Using the concept of multiplication acts various results of cancellation (weak cancellation) acts are investigated. The relation between cancellation acts and weak cancellation acts was discussed. In addition, the faithfulness concept of acts is important for some useful results in this work.

References


